A geometric approach to sector decomposition

Takahiro Ueda (U. of Tsukuba)

Based on Comput.Phys.Commun.181 (2010) 1352 (arXiv:0908.2897 [hep-ph])
with Toshiaki Kaneko (KEK)

CPP2010, KEK, 23-25 September 2010

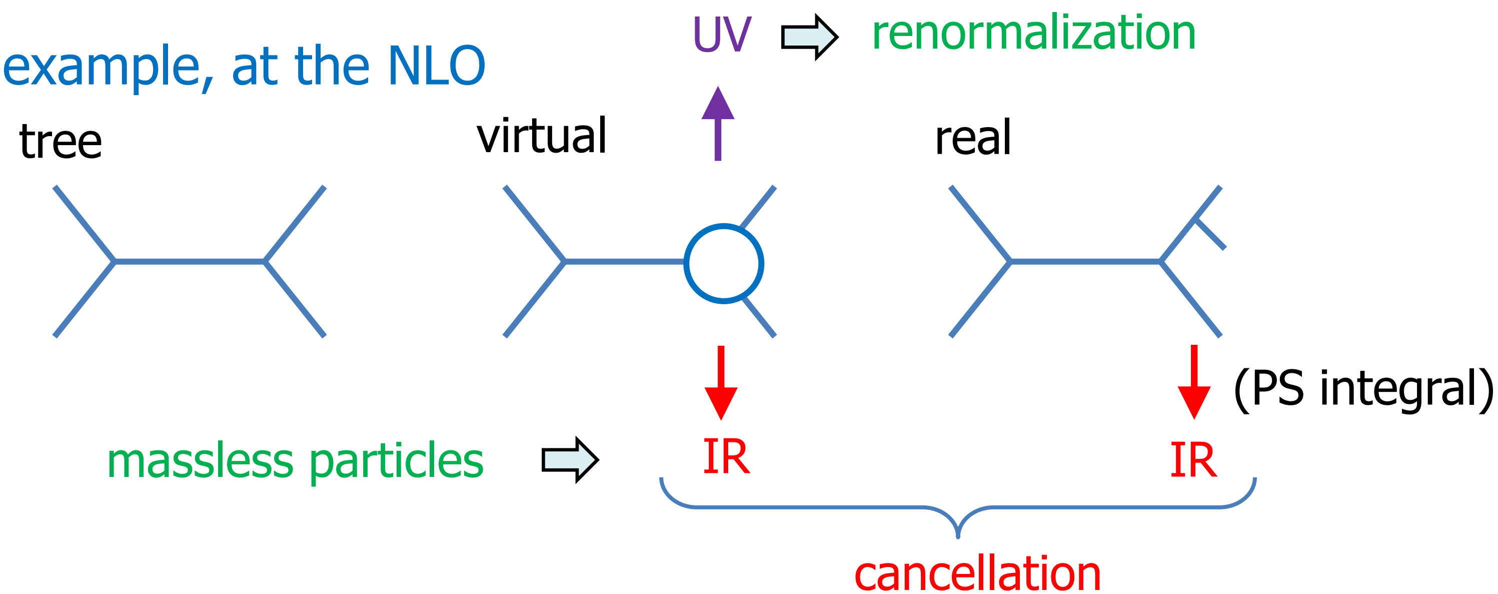
Contents

Introduction

- LHC has restarted !!

- High precision experiments require high precision theoretical calculations

For example, at the NLO



Separation of IR divergences

- NNLO, N^3LO, … ⇨ much complicated

Separation of Divergences

- One-dimensional integration example:

  if singularities are factored out as $x^{-m+n\epsilon}$

$$\int_0^1 dx \frac{f(x)}{\boxed{x^{1-\epsilon}}}$$

singular when $x \to 0$

$f(x)$: finite $\quad \epsilon \to +0$

$$= \int_0^1 dx \frac{f(0)}{x^{1-\epsilon}} + \int_0^1 dx \frac{f(x)-f(0)}{x^{1-\epsilon}}$$

Assume IR divergences are dimensionally regularized

$$= \frac{f(0)}{\boxed{\epsilon}} + \int_0^1 dx\, x^{\epsilon} \boxed{\frac{f(x)-f(0)}{x}}$$

pole

non-singular at $x \to 0$

- Question: How can we factor out divergences from multi-dimensional integration?

e.g., $$\int_0^1 dx \int_0^1 dy \frac{f(x,y)}{\boxed{(x+y)^{2-\epsilon}}}$$

singular when simultaneously $x, y \to 0$

$f(x,y)$: finite $\quad \epsilon \to +0$

- Answer: sector decomposition

T.Binoth & G.Heinrich, Nucl.Phys.B585 (2000) 741; 680 (2004) 375; 693 (2004) 134

cf. G.Heinrich, Int.J.Mod.Phys.A23 (2008) 1457

Sector Decomposition

- Sector decomposition disentangles overlapping singularities in multi-dimensional integration

$f(x,y)$: finite $\quad \epsilon \to +0$

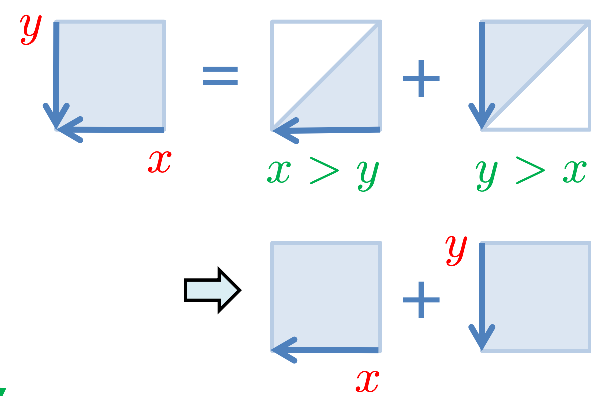
$$\int_0^1 dx \int_0^1 dy \frac{f(x,y)}{(x+y)^{2-\epsilon}}$$

singular when simultaneously $x, y \to 0$

Split integration domain

$$= \int_0^1 dx \int_0^x dy \frac{f(x,y)}{(x+y)^{2-\epsilon}} + \int_0^1 dy \int_0^y dx \frac{f(x,y)}{(x+y)^{2-\epsilon}}$$

$x > y$ $\qquad$ $y > x$



Remap to [0:1] $\quad y \to x\tilde{y}$

Similarly $\quad x \to \tilde{x}y$

$$\int_0^1 dx \int_0^1 x d\tilde{y} \frac{f(x, x\tilde{y})}{(x+x\tilde{y})^{2-\epsilon}}$$

Factor out x

$$= \int_0^1 dx \int_0^1 d\tilde{y} \frac{1}{x^{1-\epsilon}} \frac{f(x, x\tilde{y})}{(1+\tilde{y})^{2-\epsilon}} + \int_0^1 d\tilde{x} \int_0^1 dy \frac{1}{y^{1-\epsilon}} \frac{f(\tilde{x}y, y)}{(1+\tilde{x})^{2-\epsilon}}$$

singular behavior is factorized as powers of one variable

non-singular $x \to 0$

- Then poles are extracted as

$$\int_0^1 dx \frac{f(x)}{x^{1-\epsilon}} = \int_0^1 dx \frac{f(0)}{x^{1-\epsilon}} + \int_0^1 dx \frac{f(x)-f(0)}{x^{1-\epsilon}} = \frac{f(0)}{\epsilon} + \int_0^1 dx\, x^{\epsilon} \frac{f(x)-f(0)}{x}$$

singular when $x \to 0$ $\qquad$ pole $\qquad$ non-singular at $x \to 0$

Iterated Sector Decomposition

- For more complicated case

$$\int_0^1 dx_1 \cdots \int_0^1 dx_n \frac{f(x_1,\ldots,x_n)}{\left[\text{polynomial of } (x_1,\ldots,x_n)\right]^{\alpha+\beta\epsilon}}$$

singular when simultaneously some of $(x_1,\ldots,x_n) \to 0$

Iteration:
- Choose a set of $(x_i, x_j, \ldots)$ from $(x_1,\ldots,x_n)$ appropriately
- Split integration domain, e.g. $(x_i > x_j, \ldots)$, and remap to $[0,1]$

Is it trivial ?

⇨ Finally, all singularities are factored out (if iterations terminate), and one gets sum of

$$\int_0^1 d\tilde{x}_1 \cdots \int_0^1 d\tilde{x}_n \, \tilde{x}_1^{a_1+b_1\epsilon} \cdots \tilde{x}_n^{a_n+b_n\epsilon} \times \frac{f(\tilde{x}_1,\ldots,\tilde{x}_n)}{\left[(\text{const.}) + \text{polynomial of } (\tilde{x}_1,\ldots,\tilde{x}_n)\right]^{\alpha+\beta\epsilon}}$$

singular behavior is factorized

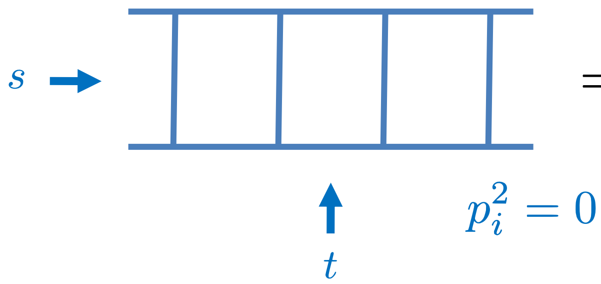
non-singular

After ϵ-expansion, coefficients of Laurent series are IR finite, and can be evaluated by Monte Carlo integration

e.g., $I = \frac{C_2}{\epsilon^2} + \frac{C_1}{\epsilon} + C_0 + \mathcal{O}(\epsilon)$

How to Split Sectors (Sector Decomposition Strategy)

- How to choose sectors to be divided; $(d = 4 - 2\epsilon)$
  i.e., choice of a set of $(x_i, x_j, \ldots)$ from $(x_1, \ldots, x_n)$



$$= 2\Gamma(4+3\epsilon) \int_0^1 d^9x \frac{\tilde{\mathcal{U}}^{2+4\epsilon}}{\tilde{\mathcal{F}}^{4+3\epsilon}}$$ + (3 terms)

massless on-shell planar triple box

$\tilde{\mathcal{U}}$ = x6*x9 + x6*x8 + x6*x7 + x5*x9 + x5*x8 + x5*x7 + x5*x6 + x4*x9 + x4*x8 + x4*x7 + x4*x6 + x3*x9 + x3*x8 + x3*x7 + x3*x6 + x3*x6*x9 + x3*x6*x8 + x3*x6*x7 + x3*x5*x9 + x3*x5*x8 + x3*x5*x7 + x3*x5*x6 + x3*x4*x9 + x3*x4*x8 + x3*x4*x7 + x3*x4*x6 + x2*x6*x9 + x2*x6*x8 + x2*x6*x7 + x2*x5*x9 + x2*x5*x8 + x2*x5*x7 + x2*x5*x6 + x2*x4*x9 + x2*x4*x8 + x2*x4*x7 + x2*x4*x6 + x2*x3*x9 + x2*x3*x8 + x2*x3*x7 + x2*x3*x6 + x1*x6*x9 + x1*x6*x8 + x1*x6*x7 + x1*x5*x9 + x1*x5*x8 + x1*x5*x7 + x1*x5*x6 + x1*x4*x9 + x1*x4*x8 + x1*x4*x7 + x1*x4*x6 + x1*x3*x9 + x1*x3*x8 + x1*x3*x7 + x1*x3*x6

$\tilde{\mathcal{F}}$ = - x6*x7*x8*s - x5*x7*x8*s - x5*x6*x7*s - x4*x7*x8*s - x4*x6*x8*s - x4*x5*x9*s - x4*x5*x8*s - x4*x5*x7*s - x4*x5*x6*s - x3*x7*x8*s - x3*x6*x9*t - x3*x6*x7*x8*s - x3*x5*x7*x8*s - x3*x5*x6*x7*s - x3*x4*x7*x8*s - x3*x4*x6*x8*s - x3*x4*x5*x9*s - x3*x4*x5*x8*s - x3*x4*x5*x7*s - x3*x4*x5*x6*s - x2*x6*x7*x8*s - x2*x5*x7*x8*s - x2*x5*x6*x7*s - x2*x4*x7*x8*s - x2*x4*x6*x8*s - x2*x4*x5*x9*s - x2*x4*x5*x8*s - x2*x4*x5*x7*s - x2*x4*x5*x6*s - x2*x3*x7*x8*s - x2*x3*x6*x7*s - x2*x3*x4*x9*s - x2*x3*x4*x8*s - x2*x3*x4*x7*s - x2*x3*x4*x6*s - x1*x6*x7*x8*s - x1*x5*x7*x8*s - x1*x5*x6*x7*s - x1*x4*x7*x8*s - x1*x4*x6*x8*s - x1*x4*x5*x9*s - x1*x4*x5*x8*s - x1*x4*x5*x7*s - x1*x4*x5*x6*s - x1*x3*x7*x8*s - x1*x3*x6*x8*s - x1*x3*x5*x9*s - x1*x3*x5*x8*s - x1*x3*x5*x7*s - x1*x3*x5*x6*s - x1*x2*x6*x9*s - x1*x2*x6*x8*s - x1*x2*x6*x7*s - x1*x2*x5*x9*s - x1*x2*x5*x8*s - x1*x2*x5*x7*s - x1*x2*x5*x6*s - x1*x2*x4*x9*s - x1*x2*x4*x8*s - x1*x2*x4*x7*s - x1*x2*x4*x6*s - x1*x2*x3*x9*s - x1*x2*x3*x8*s - x1*x2*x3*x7*s - x1*x2*x3*x6*s

Is it trivial…? **NO**, especially in higher orders

of Sectors Heavily Depends on How SD is Performed !!

- # of generated sectors by different strategies

guaranteed to terminate (A, B, C, S) — not guaranteed to terminate (X)

Diagram	A	B	C	S	X
Box	12	12	12	12	12
Double box	755	586	586	362	293
Triple box	M	114256	114256	22657	10155
D420	8898	564	564	180	F

Table in A.V. Smirnov & M.N. Tentyukov, Comput.Phys.Commun.180 (2009) 735

M: Memory overflow (8GB)

F: Failed

A: based on work of Zeillinger (2005)
B: based on work of Spivakovsky (1983)
C: Bogner & Weinzierl (2008)
S: A.Smirnov (2008) SS: Smirnov's (2009)
X: heuristic strategy; e.g., Binoth & Heinrich (2000)

- # of sectors = efficiency of Monte Carlo integration

We want a method to sector decomposition which
- is guaranteed to terminate
- gives small a number of sectors as possible

Geometric Method: Overview

- Consider a polynomial T.Kaneko & TU, Comput.Phys.Commun.181 (2010) 1352

$$Q(x_1, x_2) = x_1 x_2^4 + x_1^2 x_2^2 + x_1^4 x_2$$

and the following sector decomposition

$$\int_0^1 d^2x \left[Q(x_1, x_2)\right]^\beta = \int_0^1 d^2x (x_1 x_2^4 + x_1^2 x_2^2 + x_1^4 x_2)^\beta$$

vanishes when $x_1, x_2 \to 0$

$$\overset{\Rightarrow}{\text{SD}} \quad \sum \int_0^1 d^2x \, x_1^{a_1} x_2^{a_2} (1 + \cdots)^\beta$$

(our goal)

singular behavior is factorized

nonzero at $x_1, x_2 \to 0$

Let's see that sector decomposition can be interpreted as a set of problems in geometry

Geometric Method: Overview (cont'd)

- Key observation

$Q = x_1 x_2^4 + x_1^2 x_2^2 + x_1^4 x_2$

$$\int_0^1 d^2x = \int_0^1 d^2x\, \theta(x_1 x_2^4 > x_1^2 x_2^2, x_1^4 x_2) + \int_0^1 d^2x\, \theta(x_1^2 x_2^2 > x_1 x_2^4, x_1^4 x_2) + \int_0^1 d^2x\, \theta(x_1^4 x_2 > x_1 x_2^4, x_1^2 x_2^2)$$

Region I: 1st. term is dominant

Region II: 2nd. term is dominant

Region III: 3rd. term is dominant

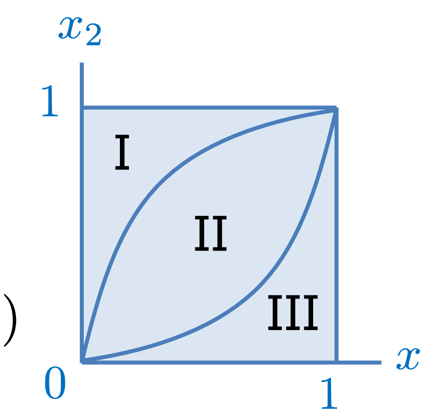
In Region I:

$$\int_0^1 d^2x (\underline{x_1 x_2^4} + x_1^2 x_2^2 + x_1^4 x_2)^\beta \times \theta(x_1 x_2^4 > x_1^2 x_2^2, x_1^4 x_2)$$

$$= \int_0^1 d^2x\, x_1^\beta x_2^{4\beta} \left(1 + \frac{x_1^2 x_2^2}{x_1 x_2^4} + \frac{x_1^4 x_2^2}{x_1 x_2^4}\right)^\beta \times \theta(x_1 x_2^4 > x_1^2 x_2^2, x_1^4 x_2)$$

singular behavior is factorized

finite at $x_1, x_2 \to 0$



Then, proper transform of integration variables

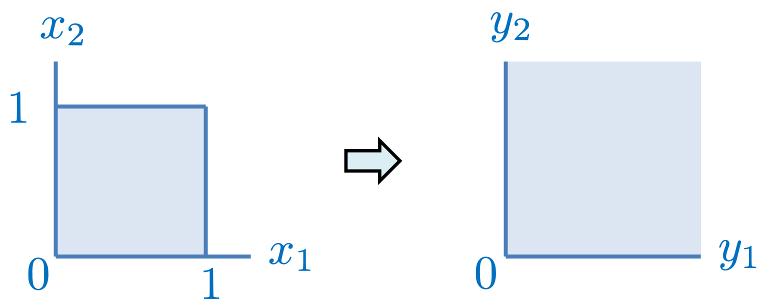
- Sector decomposition can be done by
  - 1. Finding/splitting regions in which each term is dominant
  - 2. Finding proper transform of integration variables for each region

Note: No iterations

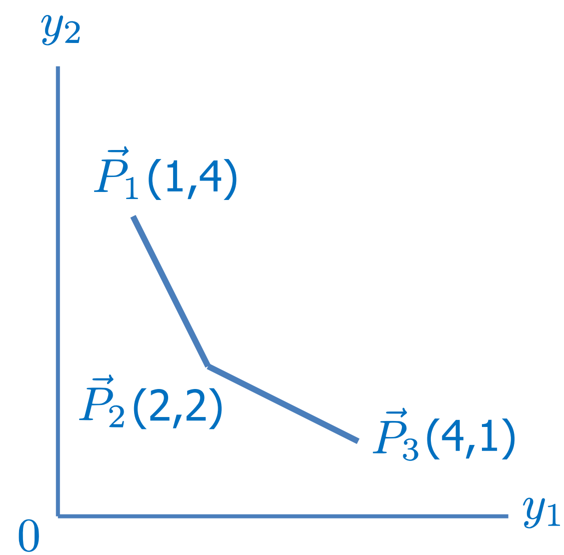
Geometric Method: Step 1 (1/3)

- 1. Finding/splitting regions in which each term is dominant

- Change variable $x_i = e^{-y_i}$



$x_i \to 0 \Longleftrightarrow y_i \to \infty$



$$
\begin{aligned}
Q &= x_1 x_2^4 + x_1^2 x_2^2 + x_1^4 x_2 \\
&= e^{-(y_1+4y_2)} + e^{-(2y_1+2y_2)} + e^{-(4y_1+y_2)} \\
&\equiv e^{-\vec{P}_1\cdot\vec{y}} + e^{-\vec{P}_2\cdot\vec{y}} + e^{-\vec{P}_3\cdot\vec{y}}
\end{aligned}
$$

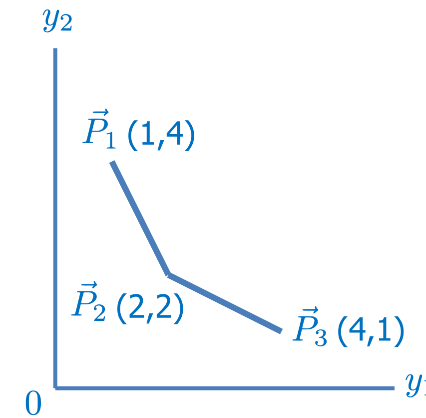
$$
\vec{P}_1 = (1,4) \quad \vec{P}_2 = (2,2) \quad \vec{P}_3 = (4,1) \quad \vec{y} = (y_1, y_2)
$$

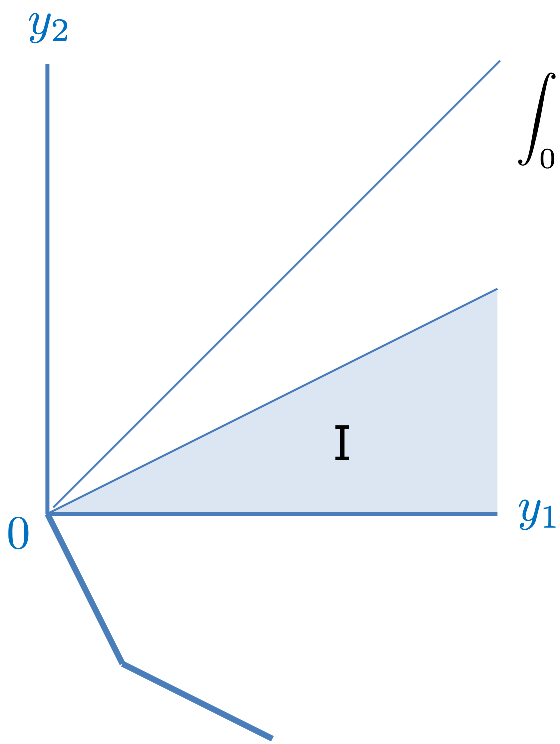
- Then 1st. term is dominant $e^{-\vec{P}_1\cdot\vec{y}} > e^{-\vec{P}_2\cdot\vec{y}}, e^{-\vec{P}_3\cdot\vec{y}}$ at $y_i \to \infty$

if $\vec{P}_1 \cdot \vec{y} < \vec{P}_2 \cdot \vec{y}, \vec{P}_3 \cdot \vec{y}$ or $\begin{cases} (\vec{P}_2 - \vec{P}_1)\cdot\vec{y} > 0 \\ (\vec{P}_3 - \vec{P}_1)\cdot\vec{y} > 0 \end{cases}$

Geometric Method: Step 1 (2/3)

- Region I: $(\vec{P}_2 - \vec{P}_1)\cdot\vec{y} > 0$ and $(\vec{P}_3 - \vec{P}_1)\cdot\vec{y} > 0$





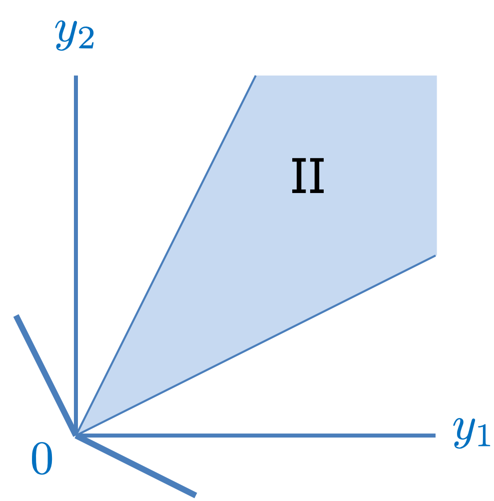
$$\int_0^1 d^2x\,\left[Q(x_1,x_2)\right]^\beta$$

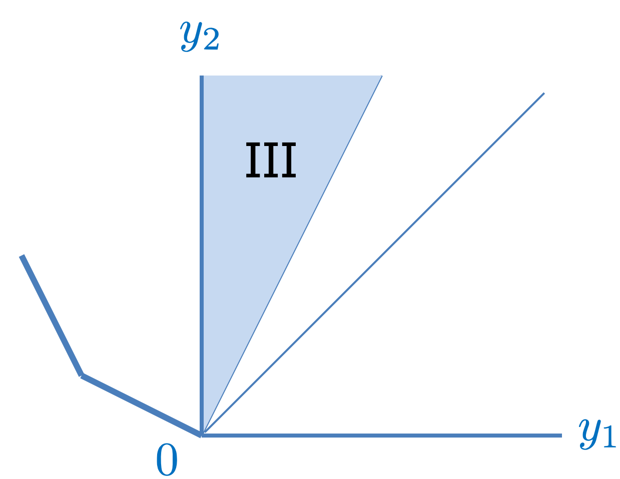
$$= \int_0^\infty d^2y\, e^{-\vec{1}\cdot\vec{y}} \left(e^{-\vec{P}_1\cdot\vec{y}} + e^{-\vec{P}_2\cdot\vec{y}} + e^{-\vec{P}_3\cdot\vec{y}}\right)^\beta$$

$$\Rightarrow \int_0^\infty d^2y\, e^{-(\beta\vec{P}_1+\vec{1})\cdot\vec{y}} \left[1 + \underbrace{e^{-(\vec{P}_2-\vec{P}_1)\cdot\vec{y}} + e^{-(\vec{P}_3-\vec{P}_1)\cdot\vec{y}}}_{\text{finite}}\right]^\beta \times \theta(\vec{y}\in\Delta_I)$$

$\vec{1} = (1,1)$

Similarly, Region II and III can be found as



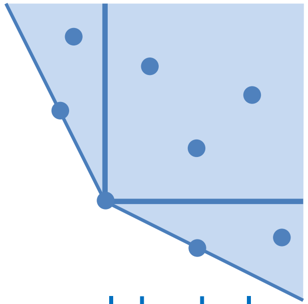


Geometric Method: Step 1 (3/3)

- More generally

  n-variable polynomial Q

  ⇨ points in n-dimensional space $Z^P = (\vec{P}_1, \vec{P}_2, \ldots)$



convex polyhedral cone

- Region in which i-th term is dominant is

$$\Delta_i^P = \left\{ \vec{y} \in \mathbb{R}^n_{\geq 0} \middle| (\vec{P}_j - \vec{P}_i) \cdot \vec{y} \geq 0, \forall \vec{P}_j \in Z^P \right\}$$

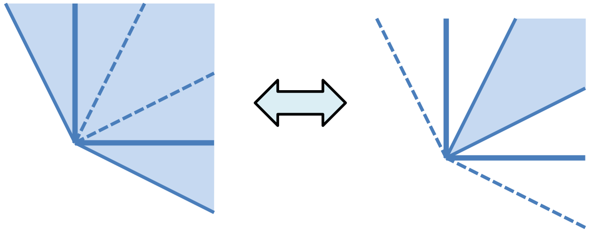
- We can construct it by using knowledge of computational geometry

  Convex polyhedral cone for a finite set S:

$$C(S) := \left\{ \sum_{\vec{v} \in S} r_{\vec{v}} \vec{v} \in \mathbb{R}^n \middle| r_{\vec{v}} \geq 0, \forall \vec{v} \in S \right\}$$

  Dual cone of convex polyhedral cone C:

$$C(S)^\vee := \left\{ \vec{y} \in \mathbb{R}^n \middle| \vec{v} \cdot \vec{y} \geq 0, \forall \vec{v} \in C(S) \right\}$$



dual cone

- In our case:

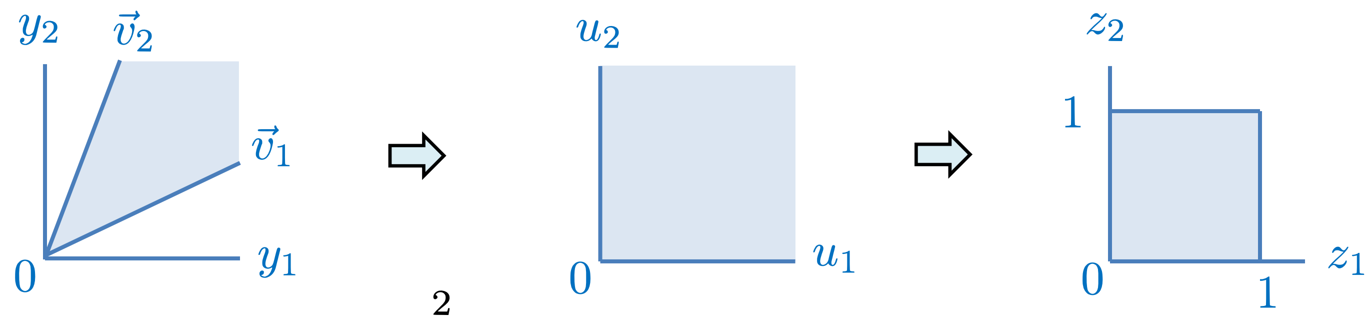
$$\Delta_i^P = C(Z_i^P)^\vee \cap \mathbb{R}^n_{\geq 0} \qquad Z_i^P = (\vec{P}_1 - \vec{P}_i, \vec{P}_2 - \vec{P}_i, \ldots)$$

  These objects are uniquely obtained by algorithms in computational geometry

Geometric Method: Step 2 (1/2)

- 2. Finding proper transform of integration variables for each region

- For 2-dimensional space



$$y_i = \sum_{j=1}^{2} (\vec{v}_j)_i u_j \qquad z_i = e^{-u_i}$$

Then we complete sector decomposition

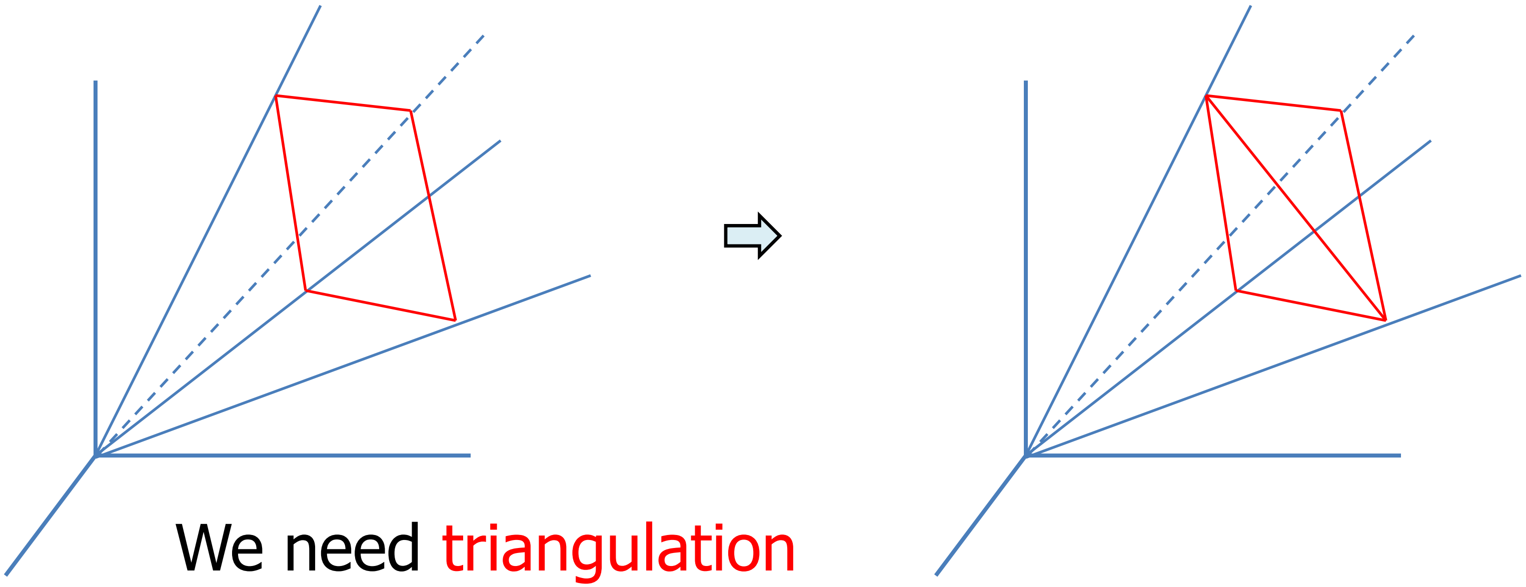
$$\int_0^\infty d^2y\, e^{-(\beta \vec{P}_1 + \vec{1})\cdot\vec{y}} \left[1 + e^{-(\vec{P}_2 - \vec{P}_1)\cdot\vec{y}} + e^{-(\vec{P}_3 - \vec{P}_1)\cdot\vec{y}}\right]^\beta \times \theta(\vec{y} \in \Delta_I)$$

$$= |\det V| \int_0^1 d^2z \prod_{i=1}^{2} z_i^{(\beta\vec{P}_1+\vec{1})\cdot\vec{v}_i - 1} \left[1 + \prod_{i=1}^{2} z_i^{(\vec{P}_2-\vec{P}_1)\cdot\vec{v}_i} + \prod_{i=1}^{2} z_i^{(\vec{P}_3-\vec{P}_1)\cdot\vec{v}_i}\right]^\beta$$

$V = (\vec{v}_1, \vec{v}_2)$ For region I, Δ_I : $\vec{v}_1 = (3,0)$ $\vec{v}_2 = (2,1)$

Geometric Method: Step 2 (2/2)

- 3- or higher dimensional spaces, regions are not simplex



We need triangulation

- Then, perform transform of integration variables:

$$y_i = \sum_{j=1}^{n} (\vec{v}_j)_i u_j \qquad z_i = e^{-u_i}$$

- Note: How to triangulate polyhedral cones (and # generated simplexes) is not unique

Test Implementation

- Our test implementation
  - Dual cone construction
    - Convex hull - Incremental algorithm
  - Triangulation
    - Our own simple algorithm; Neither unique, nor optimum
  - Written in Python (and very slow)

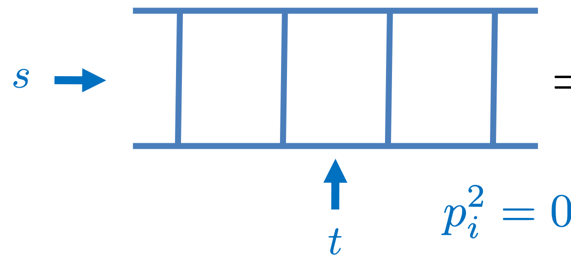
# of generated sectors

Diagram	A	B	C	S	X	our method
Box	12	12	12	12	12	12
Double box	755	586	586	362	293	266
Triple box	M	114256	114256	22657	10155	6568
D420	8898	564	564	180	F	168

Our method gives less number of sectors than other methods !!

An Example Calculation

- On-shell triple box



$$= \Gamma(4+3\epsilon)\left[\sum_{i=0}^{6}\frac{C_i}{\epsilon^i} + \mathcal{O}(\epsilon)\right]$$

Taking symmetries into account, Strategy X and our method generate

Strategy X : 4143 terms
Out method : 2670 terms
(ratio $\sim$ 1.55)

	Strategy X		Our method		Exact value
C_6	0.098765(±0.000004)	1.4s	0.098765(±0.000003)	0.7s	0.098765
C_5	-0.767001(±0.000027)	10.7s	-0.767001(±0.000033)	4.4s	-0.767001
C_4	1.977227(±0.000156)	2.9m	1.977224(±0.000326)	1.1m	1.977227
C_3	-0.753089(±0.001144)	26.6m	-0.753088(±0.001192)	10.1m	-0.753080
C_2	-4.749675(±0.006682)	2.5h	-4.749694(±0.003523)	1.3h	-4.749610
C_1	2.016040(±0.042107)	6.9h	2.016607(±0.014341)	4.1h	2.016790
C_0	21.673992(±0.268047)	16.4h	21.692735(±0.067518)	9.9h	21.692455

s = -1, t = -3, integrated by Cuba1.7 (Vegas) 10Mpts, Xeon X5560 (2.80GHz) x 8cores

16.4h / 9.9h $\sim$ 1.66

Analytical result: V.A. Smirnov, Phys.Lett.B567 (2003) 193

Summary

- Sector decomposition is used for the separation of divergences from multi-dimensional integrations

- # of generated sectors = efficiency of Monte Carlo integration

- We have proposed a new method of sector decomposition employing a geometric interpretation of the problem
  - Regions in which each term is dominant – Dual cone construction
  - Proper transform of integration variables – Triangulation

    (No iteration)

  They are solved by using algorithms in computational geometry

- A test implementation showed the smaller number of sectors compared to other method based on iterated decomposition

Appendix

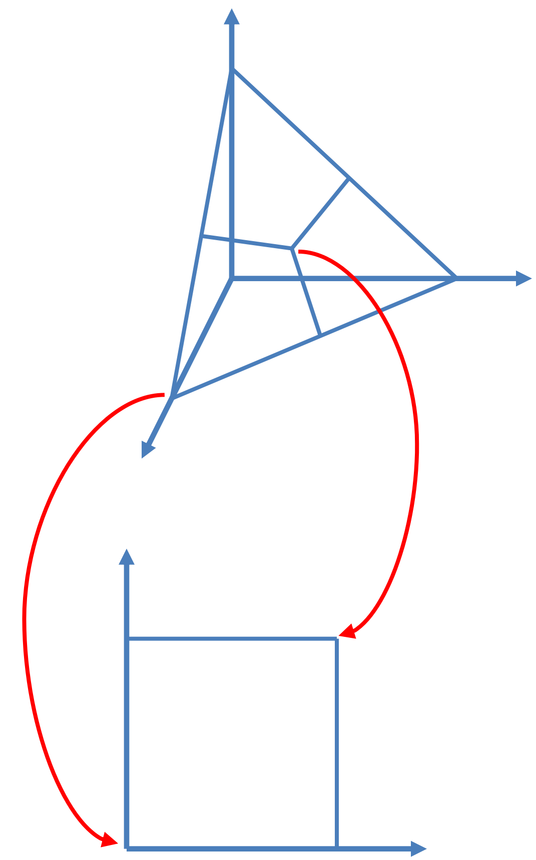
Feynman Parameter Representation

$$G = (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^\infty d^N x \, x^{\nu - \mathbb{I}} \delta \left(1 - \sum_{l=1}^{N} x_l \right) \frac{\mathcal{U}^{N_\nu - (L+1)D/2}}{\mathcal{F}^{N_\nu - LD/2}}$$

- L is the number of loops,
- $D = 4 - 2\epsilon$ is the dimension of the space-time,
- N is the dimension of the integration,
- ν_j $(j = 1, \cdots, N)$ is the power of the propagator corresponding to the Feynman parameter x_j,
- $N_\nu := \sum_{j=1}^{N} \nu_j$,
- $\mathcal{U}$ is a homogeneous polynomial of $\{x_j\}$ of degree L, and all the coefficients of the monomials of $\mathcal{U}$ are equal to 1.
- $\mathcal{F}$ is a homogeneous polynomial of $\{x_j\}$ of degree $L+1$, and the coefficients of the monomials of $\mathcal{F}$ consist of kinematic and mass parameters.

Primary Sector Decomposition

T.Binoth & G.Heinrich, Nucl.Phys.B585 (2000) 741



- (Sector l) = $\{(x_1, x_2, \cdots, x_N) \mid x_j \leq x_l, \ \forall j \neq l\}$
- Change variables

$$x_j = x_l \times \begin{cases} t_j & \text{for} \quad j < l \\ 1 & \text{for} \quad j = l \\ t_{j-1} & \text{for} \quad j > l \end{cases}$$

- Integration over x_l with δ -function
- Results

$$G = (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \times \sum_{l=1}^{N} \int_0^1 d^{N-1}t \, t^{\nu' - \mathbb{I}} \, \mathcal{U}_l^{\gamma}(t) \, \mathcal{F}_l^{\beta}(t)$$

$$\beta = -(N_\nu - LD/2) \quad \gamma = N_\nu - (L+1)D/2$$

- IR singularities: boundary ⇨ some of variables $t_j = 0$

How Sector Decomposition Works

- Concrete example: 1-loop scalar massless on-shell box

$p_i^2 = 0 \quad (d = 4 - 2\epsilon)$

$$
\begin{aligned}
I_4^{0m} &= \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^2(k+p_1)^2(k+p_1+p_2)^2(k+p_1+p_2+p_3)^2} \\
&= 2\Gamma(2+\epsilon)\int_0^1 d^3x \, \frac{(1+x_1+x_2+x_3)^{2\epsilon}}{(-sx_1x_3-tx_2)^{2+\epsilon}} + (s\leftrightarrow t) \\
&= 2\Gamma(2+\epsilon)\int_0^1 d^3x \Bigg[x_1^{-1-\epsilon}\frac{(1+x_1+x_2+x_1x_3)^{2\epsilon}}{(-sx_1x_3-t)^{2+\epsilon}} \\
&\qquad + x_1^{-1-\epsilon}x_2^{-1-\epsilon}\frac{(1+x_1+x_1x_2+x_2x_3)^{2\epsilon}+(1+x_1+x_2+x_1x_2x_3)^{2\epsilon}}{(-sx_3-t)^{2+\epsilon}}\Bigg] + (s\leftrightarrow t) \\
&= \Gamma(2+\epsilon)\left[\frac{C_2}{\epsilon^2} + \frac{C_1}{\epsilon} + C_0 + \mathcal{O}(\epsilon)\right]
\end{aligned}
$$

Feynman parameter

Iterated sector decomposition

separation of poles $1/\epsilon$

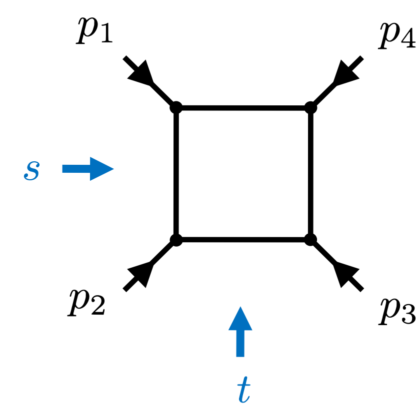
ϵ-expansion

where

$$
C_2 = 4\int_0^1 dx \, \frac{1}{(-sx-t)^2} + (s\leftrightarrow t)
$$

$$
C_1 = \left[-4\int_0^1 dx \, \frac{\ln(-sx-t)}{(-sx-t)^2} - 2\int_0^1 d^2x \, \frac{1}{(-sx_1x_2-t)^2}\right] + (s\leftrightarrow t)
$$

$$
C_0 = \left\{2\int_0^1 dx \, \frac{\ln^2(-sx-t)}{(-sx-t)^2} + 2\int_0^1 d^2x \left[\frac{\ln(-sx_1x_2-t) - 2\ln(1+x_2)}{(-sx_1x_2-t)^2} - \frac{2}{(-sx_2-t)}\frac{\ln(1+x_1x_2)+3\ln(1+x_1)}{x_1}\right]\right\} + (s\leftrightarrow t)
$$

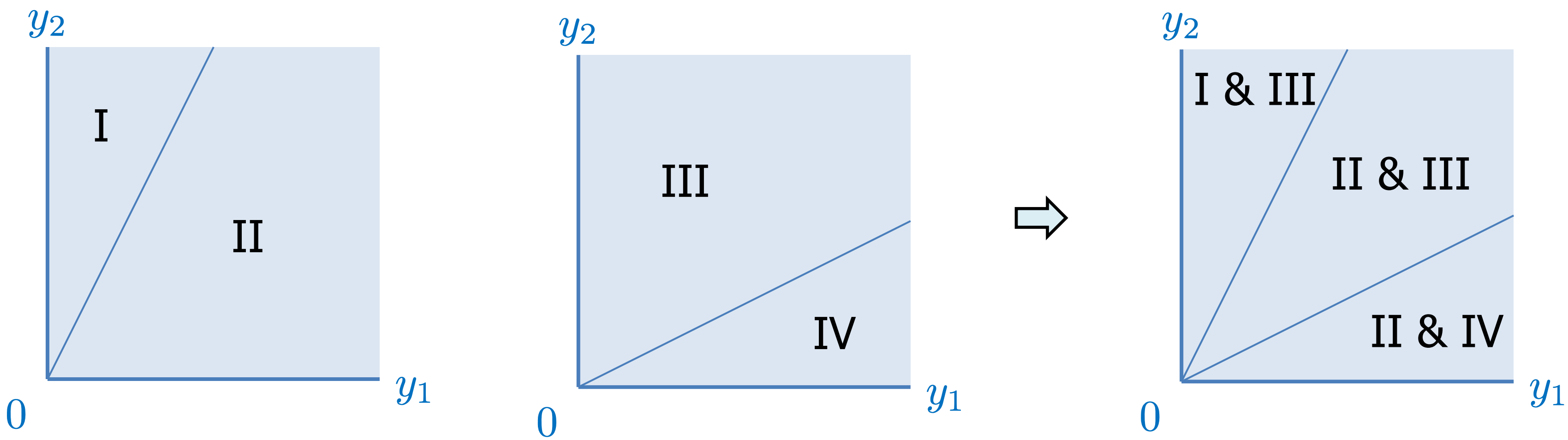


- Coefficients of Laurent series can be evaluated (numerically)

Two or More Polynomials

- Product of polynomials $Q_1^{\beta_1} Q_2^{\beta_2}$

Q_1 : Q_2 :



Take intersections

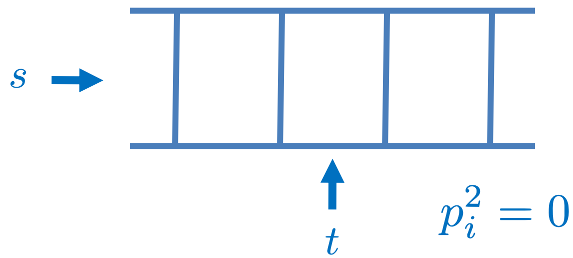
- identity

$$\Delta = C(Z_1^P)^\vee \cap C(Z_2^P)^\vee \cap \mathbb{R}^n_{\geq 0} = C(Z_1^P \cup Z_2^P \cup E^n)^\vee$$

$E^n =$ (the set of unit vectors)

An Example Calculation

- Triple Box



$$= \Gamma(4+3\epsilon)\left[\sum_{i=0}^{6} \frac{C_i}{\epsilon^i} + \mathcal{O}(\epsilon)\right]$$

Taking symmetries into account, Strategy X and our method generate

Strategy X	:	4143 terms	(ratio ~ 1.55)
Out method	:	2670 terms	

Size of binaries and timings:

	Strategy X		Our method	
C_6	52K	1.4s	45K	0.7s
C_5	160K	10.7s	86K	4.4s
C_4	1.3M	2.9m	541K	1.1m
C_3	6.9M	26.6m	3.3M	10.1m
C_2	26M	2.5h	14M	1.3h
C_1	72M	6.9h	42M	4.1h
C_0	176M	16.4h	105M	9.9h

176M / 105M ~ 1.68

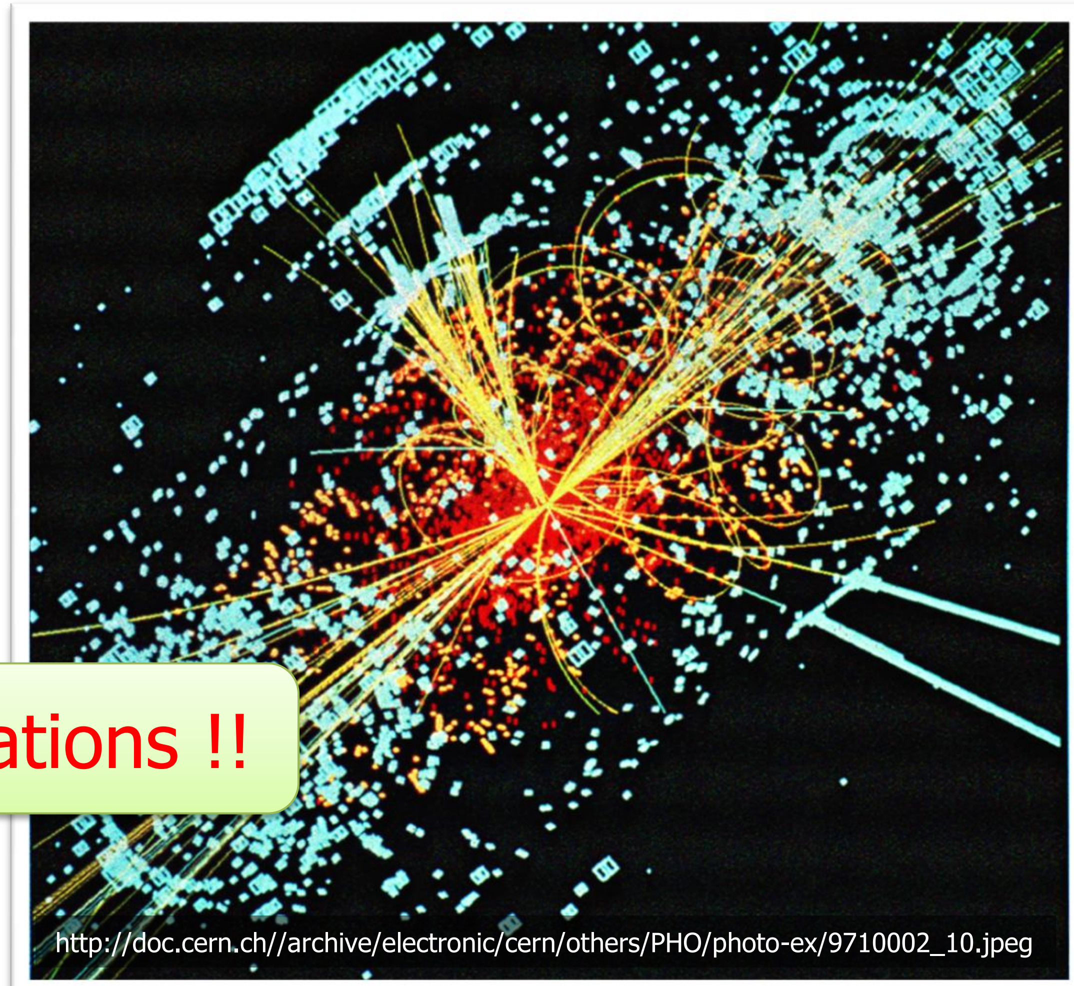
16.4h / 9.9h ~ 1.66

s = -1, t = -3, integrated by Cuba1.7 (Vegas) 10Mpts, Xeon X5560 (2.80GHz) x 8cores

Backup

Life in the Collider Era

- LHC = Large Hadron Collider
  - Signals on large backgrounds
  - needs precise theoretical predictions
  - ILC also needs lots of precision calculations



Higher order calculations !!