A new method to compute the one loop scalar integrals

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Original idea of Shimizu-san

A scalar two loop three point function can be written as

$$\sum \int_0^1 d\xi \int_0^1 d\rho \, \tilde{I}_4^4(\xi, \rho)$$

A scalar two loop four point function can be written as

$$\sum \int_0^1 d\xi \int_0^1 d\rho \, \tilde{I}_5^4(\xi, \rho)$$

$\tilde{I}_4^4$: pseudo scalar one loop four point function
$\tilde{I}_5^4$: pseudo scalar one loop five point function

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pseudo means that the volume defined by the Feynman parameter integral is not the usual simplex but it can be an hyper-cube for example

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As $I_5^4 = \sum_i c_i I_4^4$, one could expect that scalar two loop three and four point functions be calculated (numerically) as a two dimensional integral of four point functions.

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Apart the problem of the volume, we can not use the usual four point function library (FF, looptools, ...) because the internal kinematic depends on ξ and ρ, so there is no guarantee that we are in the physical domain.

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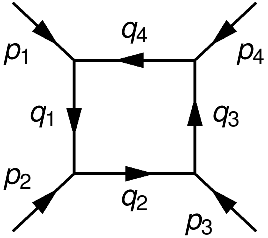
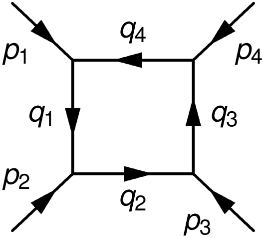
Can we get rid of that?



$$I_4^4 = \int_0^1 \prod_{i=1}^{4} dx_i \, \delta(1 - \sum_{i=1}^{4} x_i) \left(-\frac{1}{2} X^T \mathcal{S} X - i\lambda \right)^{-2}$$

$$\mathcal{S}_{ij} = (q_i - q_j)^2 - m_i^2 - m_j^2, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix}$$

Four-point function
Definition



$$I_4^4 = \int_0^1 \prod_{i=1}^{4} dx_i \, \delta(1 - \sum_{i=1}^{4} x_i) \left(-\frac{1}{2} X^T \mathcal{S} X - i \lambda \right)^{-2}$$

Assumption

The imaginary part of the denominator keeps a constant sign in the simplex

Definition for $i, j \neq a$

$$\begin{aligned} G_{ij}^{(a)} &= -(\mathcal{S}_{ij} - \mathcal{S}_{aj} - \mathcal{S}_{ia} + \mathcal{S}_{aa}) \\ V_j^{(a)} &= \mathcal{S}_{aj} - \mathcal{S}_{aa} \end{aligned}$$

single out x_4 :

$$I_4^4 = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \, \frac{1}{(D(X) - i\,\lambda)^2}$$

with

$$D(X) = X^T.A.X + B^T.X + C, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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\begin{aligned}
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\end{aligned}
$$

$$
D(X) = X^T.A.X + B^T.X + C, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

and where

$$
A_{ij} = \frac{1}{2} G_{ij}^{(4)}, \quad B_i = -V_i^{(4)}, \quad C = -\frac{1}{2}\mathcal{S}_{44}
$$

Extensive use of the identity

$$\frac{1}{D^{\alpha+1}} = \frac{1}{\alpha\,\Delta_n}\left[2\,\frac{n-2\,\alpha}{D^{\alpha}} - \nabla^{T}.\left(\frac{2\,X + A^{-1}.B}{D^{\alpha}}\right)\right]$$

n : the number of independent integration variables

$$\Delta_n = B^T.A^{-1}.B - 4\,C$$

The idea is to adjust the power of the denominator in the l.h.s in such way that only the boundary term remains.

In the case of the four point function $n = 3$.

$$n - 2\alpha = 0 \to \alpha = 3/2, \quad \text{i. e.} \quad \alpha + 1 = 5/2$$

However in four point function, D is raised to the power 2 not 5/2.

To shift the power of the denominator

$$\int_0^\infty \frac{d\xi}{(D + \xi^\nu)^\mu} = \frac{1}{\nu} B\left(\frac{1}{\nu}, \mu - \frac{1}{\nu}\right) \frac{1}{D^{\mu - 1/\nu}} \tag{1}$$

where $B(x, y)$ is the Euler beta function defined by

$$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x + y)}$$

This integral is convergent for $\mu > 1/\nu > 0$.

$$
\begin{aligned}
I_4^4 \;=\; & \frac{1}{3\,B(2,1/2)} \int_0^{\infty} d\xi \, \frac{1}{\xi^2 - \Delta_3/4 \, - i\,\lambda} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \\
& \times \int_0^{1-x_1-x_2} dx_3 \, \nabla^T . \left(\frac{2\,X + A^{-1}.B}{(D(x_1,x_2,x_3) + \xi^2 - i\,\lambda)^{3/2}} \right) \qquad (2)
\end{aligned}
$$

with $\Delta_3 = B^T.A^{-1}.B - 4\,C$.

$$I_4^4 = \frac{1}{3\,B(2,1/2)} \int_0^\infty d\xi \, \frac{1}{\xi^2 - \Delta_3/4 \, - i\,\lambda} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \times \int_0^{1-x_1-x_2} dx_3 \, \nabla^T . \left(\frac{2\,X + A^{-1}.B}{(D(x_1,x_2,x_3) + \xi^2 - i\,\lambda)^{3/2}} \right) \quad (2)$$

with $\Delta_3 = B^T.A^{-1}.B - 4\,C$.

Warning

We have to be careful because we are working with complex numbers! Is this integral well defined? What happened if $\mathrm{Im}(\Delta_3) \to 0$?

Residue in ξ

$$R = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \, \nabla^T . \left(\frac{2\,X + A^{-1}.B}{(D(X) + \Delta_3/4)^{3/2}} \right)$$

If $\mathrm{Im}(\Delta_3) = 0$, the imaginary part of the denominator is driven by $D(X)$ which is of the same sign in the simplex.

$$\text{As} \quad \nabla^T \left(\frac{2\,X + A^{-1}.B}{(D(X) + \Delta_3/4)^{3/2}} \right) = 0 \quad \text{for a gentle function}$$

So in this case $R = 0$

$$I_4^4 = -\frac{2}{3}\,\frac{1}{B(2,1/2)} \int_0^{\infty} d\xi \, \frac{1}{\xi^2 - \Delta_3/4 - i\,\lambda} \sum_{i=1}^{4} \frac{\bar{b}_i}{\det{(G)}} \times \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \, \frac{1}{(\overline{D}_i(x_1,x_2) + \xi^2 - i\,\lambda)^{3/2}}$$

with

$$\Delta_3 = -2\,\frac{\det{(\mathcal{S})}}{\det{(G)}}$$

$$\bar{b}_i = \det{(\mathcal{S})} \sum_{j \in S} \mathcal{S}^{-1}_{i\,j}$$

The new quadratic forms $\overline{D}_i(x_1, x_2)$ are defined by:

$$\overline{D}_1(x_1, x_2) = D(0, x_1, x_2) \quad \overline{D}_2(x_1, x_2) = D(x_2, 0, x_1)$$
$$\overline{D}_3(x_1, x_2) = D(x_1, x_2, 0) \quad \overline{D}_4(x_1, x_2) = D(1 - x_1 - x_2, x_1, x_2)$$

$$\overline{D}_i(x_1, x_2) = \tilde{X}^{\,T}.E_i.\tilde{X} + F_i^T.\tilde{X} + G_i \quad , \quad \tilde{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

One can iterate the method twice ...

$$
\begin{aligned}
I_4^4 &= \frac{4}{3}\,\frac{1}{B(2,1/2)}\,\frac{1}{B(3/2,1/2)}\,\frac{1}{B(1,1/2)} \sum_{i \in S}\ \sum_{j \in S\setminus\{i\}}\ \sum_{k \in S\setminus\{i,j\}} \\
&\times \frac{\bar{b}_i}{\det(G)}\,\frac{\bar{b}_j^{\{i\}}}{\det(G^{\{i\}})}\,\frac{\bar{b}_k^{\{ij\}}}{\det(G^{\{ij\}})} \int_0^\infty d\xi \int_0^\infty d\rho \int_0^\infty d\sigma \\
&\times \frac{1}{\xi^2 - \Delta_3/4 - i\,\lambda}\,\frac{1}{\xi^2 + \rho^2 - \Delta_2^i/4 - i\,\lambda} \\
&\times \frac{1}{\xi^2 + \rho^2 + \sigma^2 - \Delta_1^{ij}/4 - i\,\lambda}\,\frac{1}{(\tilde{D}_{ijk} + \xi^2 + \rho^2 + \sigma^2 - i\,\lambda)^{1/2}}
\end{aligned}
$$

$\mathcal{S}^{\{i\}}$: a 3×3 matrix made from $\mathcal{S}$ by removing the row and column i

$$\tilde{D}_{ijk} = m_l^2 \quad \text{with } l = S \setminus \{i\,j\,k\}$$

$$\Delta_1^{ij} = -\frac{2\,\det\left(\mathcal{S}^{\{i,j\}}\right)}{\det\left(G^{\{i,j\}}\right)}$$

$$\Delta_2^{i} = \frac{2\,\det\left(\mathcal{S}^{\{i\}}\right)}{\det\left(G^{\{i\}}\right)}$$

$$
\begin{aligned}
L_{ijk} &= -\frac{\kappa}{4^3} \int_0^\infty d\xi \int_0^\infty d\rho \int_0^\infty d\sigma \\
&\times \frac{1}{\xi^2 - \frac{\Delta_3}{4} - i\,\lambda} \frac{1}{\xi^2 + \rho^2 - \frac{\Delta_2^i}{4} - i\,\lambda} \\
&\times \frac{1}{\xi^2 + \rho^2 + \sigma^2 - \frac{\Delta_1^{ij}}{4} - i\,\lambda} \frac{1}{(\tilde{D}_{ijk} + \xi^2 + \rho^2 + \sigma^2 - i\,\lambda)^{1/2}}
\end{aligned}
$$

with

$$
\kappa = -\frac{256}{3} \frac{1}{B(2, 1/2)} \frac{1}{B(3/2, 1/2)} \frac{1}{B(1, 1/2)}
$$

$$J = \int_0^\infty d\xi \, \frac{1}{\xi^2 + A} \, \frac{1}{(\xi^2 + B)^{1/2}}$$

where A and B are complex numbers.Using Feynman parameter method, we can show that :

$$J = \begin{cases} \int_0^1 dz \, \frac{1}{z^2 \, B + (1 - z^2) \, A} & \text{if } S_A = S_B \\ i \, S_B \, \int_0^\infty \frac{dz}{B \, z^2 - (1 + z^2) \, A} - \int_1^\infty \frac{dz}{B \, z^2 + (1 - z^2) \, A} & \text{otherwise} \end{cases}$$

where $S_K = \text{sign}(\text{Im}(K))$ with $K = A, B$.

- σ integration : using preceding formula
- ρ integration : using partial fraction decomposition and eq. (1)
- ξ integration : using again the preceding formula

case $\text{Im}(\Delta_3) > 0$, $\text{Im}(\Delta_2^i) > 0$, $\text{Im}(\Delta_1^{ij} > 0$, $\text{Im}(\tilde{D}_{ijk}) < 0$

$$L_{ijk} = \frac{1}{2} \int_0^1 \frac{du}{u^2\, P_{ijk} + R_{ij}} \left[\int_0^1 \frac{dz}{z^2\,(T + Q_i) + (1 - z^2)\, T} - \int_0^1 \frac{dz}{z^2\,(u^2\, P_{ijk} + R_{ij} + Q_i + T) + (1 - z^2)\, T} \right]$$

$$
\begin{aligned}
P_{ijk} &= \tilde{D}_{ijk} + \frac{\Delta_1^{ij}}{4} \\
Q_i &= \frac{\Delta_3}{4} - \frac{\Delta_2^i}{4} \\
R_{ij} &= \frac{\Delta_2^i}{4} - \frac{\Delta_1^{ij}}{4} \\
T &= -\frac{\Delta_3}{4}
\end{aligned}
$$

$$L_{ijk} = -\frac{1}{4} \int_0^1 du \, \frac{1}{u^2 \, P_{ijk} \, Q_i - R_{ij} \, T} \left[\ln \left(\frac{u^2 \, P_{ijk} + R_{ij} + Q_i + T}{u^2 \, (P_{ijk} + R_{ij} + Q_i) + T} \right) - \ln \left(\frac{Q_i + T}{u^2 \, Q_i + T} \right) \right] \tag{3}$$

8 cases to be considered !

case $\mathrm{Im}(\Delta_3) < 0$, $\mathrm{Im}(\Delta_2^i) > 0$, $\mathrm{Im}(\Delta_1^{ij}) > 0$, $\mathrm{Im}(\tilde{D}_{ijk}) < 0$

$$
\begin{aligned}
L_{ijk} = \frac{1}{4} \Bigg\{ & i \int_0^\infty du \, \frac{1}{u^2 \, P_{ijk} \, Q_i + R_{ij} \, T} \left[\ln \left(\frac{R_{ij} + Q_i}{u^2 \, (P_{ijk} + R_{ij} + Q_i) - T} \right) - \ln \left(\frac{Q_i}{Q_i \, u^2 - T} \right) \right] \\
& + \int_1^\infty du \, \frac{1}{u^2 \, P_{ijk} \, Q_i - R_{ij} \, T} \left[\ln \left(\frac{R_{ij} + Q_i}{u^2 \, (P_{ijk} + R_{ij} + Q_i) + T} \right) - \ln \left(\frac{Q_i}{Q_i \, u^2 + T} \right) \right] \\
& + \int_0^1 du \, \frac{1}{u^2 \, P_{ijk} \, Q_i - R_{ij} \, T} \left[\ln \left(\frac{R_{ij} + Q_i}{u^2 \, P_{ijk} + R_{ij} + Q_i + T} \right) - \ln \left(\frac{Q_i}{Q_i + T} \right) \right] \Bigg\}
\end{aligned}
\tag{4}
$$

For infrared divergent cases, one or several sectors have $\Delta_2^i = 0$ (sub-leading Landau singularities). For such a sector, only one $\bar{b}_i^{\{j\}} \neq 0$, $\Delta_1^{ij} \neq 0$ but two scenarios :

- $\tilde{D}_{ijk} = 0$ (soft divergence)
- $\tilde{D}_{ijk} \neq 0$ (soft and collinear divergence)

we have to compute in a $n = 4 - 2\,\varepsilon$ space time dimension

$$
\begin{aligned}
L_{ijk} &= -\frac{\kappa}{4^3} \int_0^{\infty} d\xi \int_0^{\infty} d\rho \int_0^{\infty} d\sigma \\
&\times \frac{1}{\xi^{\nu} - \frac{\Delta_3}{4} - i\,\lambda} \, \frac{1}{\xi^{\nu} + \rho^2 - \frac{\Delta_2^i}{4} - i\,\lambda} \\
&\times \frac{1}{\xi^{\nu} + \rho^2 + \sigma^2 - \frac{\Delta_1^{ij}}{4} - i\,\lambda} \, \frac{1}{(\tilde{D}_{ijk} + \xi^{\nu} + \rho^2 + \sigma^2 - i\,\lambda)^{1/2}}
\end{aligned}
$$

with $\nu = 2/(1 - 2\,\varepsilon)$

$$J = \int_0^\infty d\xi \, \frac{1}{\xi^\nu + A} \, \frac{1}{(\xi^\nu + B)^{1/2}}$$

if $S_A = S_B$:

$$K(\varepsilon) \int_0^1 dz \, \frac{1}{(z^2 \, B + (1 - z^2) \, A)^{1+\varepsilon}}$$

else :

$$K(\varepsilon) \left[i \, S_B \, e^{i \, \varepsilon \, S_B \, \pi} \int_0^\infty \frac{dz}{(B \, z^2 - (1 + z^2) \, A)^{1+\varepsilon}} - \int_1^\infty \frac{dz}{(B \, z^2 + (1 - z^2) \, A)^{1+\varepsilon}} \right]$$

with $K(\varepsilon) = \frac{1-2\,\varepsilon}{2} \, B(1/2 - \varepsilon, 1 + \varepsilon)$

Different cases depending on sign(Im(Δ_3)) and sign(Im(Δ_1^{ij})).

case Im(Δ_3) $>$ 0 and Im(Δ_1^{ij}) $>$ 0 :

$$L_{ijk} = \frac{1}{8\,(1+\varepsilon)}\,\frac{1}{R_{ij}\,T}\,\frac{(\Gamma(1-\varepsilon))^2}{\Gamma(1-2\,\varepsilon)}\,\left[\frac{1}{\varepsilon^2}\,\left(4\,R_{ij}\right)^{-\varepsilon} + \mathrm{Li}_2\left(\frac{T-R_{ij}}{T}\right) - \frac{\pi^2}{6}\right]$$

Different cases depending on sign(Im(Δ_3)), sign(Im(Δ_1^{ij})) and sign(Im($\tilde{D}_{ijk}$)).

case Im$(\Delta_3) > 0$, Im$(\Delta_1^{ij}) > 0$ and Im$(\tilde{D}_{ijk}) < 0$:

$$L_{ijk} = \frac{1}{1+\varepsilon}\,\frac{1}{4\,T}\left[-\frac{T^{-\varepsilon}}{\varepsilon}\left(\int_0^1 \frac{du}{u^2\,P_{ijk}+R_{ij}} - \varepsilon\int_0^1 du\,\frac{\ln(1-u^2)}{u^2\,P_{ijk}+R_{ij}}\right)\right.$$
$$\left. + \int_0^1 \frac{du}{u^2\,P_{ijk}+R_{ij}}\ln\left(\frac{u^2\,P_{ijk}+R_{ij}}{u^2\,(P_{ijk}+R_{ij}-T)+T}\right)\right]$$

Conclusion

Method with nice features

- it is valid outside the physical domain
- it runs smoothly with all the traps of complex mass cases
- it is expressed in term of the (reduced) kinematical matrix and the (reduced) Gram matrix

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Fortran95 code, checks with looptools

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Numerical implementation

Fortran95 code, checks with looptools

Drawback

This method generates more dilogarithms than other methods (HV), numerical instability in some phase space region ($p_i^2 = 0$ for example), need to rearrange the final formulae

Conclusion

Future work

Extend to the case where the volume of the “one loop” Feynman parameters is not a simplex