Functional equations for Feynman integrals

O.V. Tarasov

JINR, Dubna, Russia

October 9, 2016, Hayama, Japan

Contents

FE from recurrence relations

Functional equations (FE) for Feynman integrals were proposed in O.V.T. Phys.Lett. B670 (2008) 67.

Feynman integrals satisfy recurrence relations which can be written as

$$\sum_j Q_j I_{j,n} = \sum_{k,r<n} R_{k,r} I_{k,r}$$

where Q_j, R_k are polynomials in masses, scalar products of external momenta, d, and powers of propagators. $I_{k,r}$ - are integrals with r external lines. In recurrence relations some integrals are more complicated than the others: they have more arguments than the others.

General method for deriving functional equations:

By choosing kinematic variables, masses, indices of propagators remove most complicated integrals, i.e. impose conditions :

$$Q_j = 0$$

keeping at least some other coefficients $R_k \neq 0$.

Introduction

Example: one-loop n-point integrals

Integrals $I_n^{(d)}$ satisfy generalized recurrence relations O.T. in Phys.Rev.D54 (1996) p.6479

$$G_{n-1}\nu_j \mathbf{j}^+ I_n^{(d+2)} - (\partial_j \Delta_n) I_n^{(d)} = \sum_{k=1}^{n} (\partial_j \partial_k \Delta_n) \mathbf{k}^- I_n^{(d)},$$

where $\mathbf{j}^{\pm}$ shifts indices $\nu_j \to \nu_j \pm 1$, $\partial_j \equiv \frac{\partial}{\partial m_j^2}$,

$$G_{n-1} = -2^n \begin{vmatrix} p_1 p_1 & p_1 p_2 & \dots & p_1 p_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_1 p_{n-1} & p_2 p_{n-1} & \dots & p_{n-1} p_{n-1} \end{vmatrix},$$

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix}, \quad Y_{ij} = m_i^2 + m_j^2 - p_{ij}, \qquad p_{ij} = (p_i - p_j)^2,$$

One-loop propagator type integral

At $n=3,\ j=1,\ m_3^2=0$, imposing conditions $G_2=0,\ \Delta_3=0$ we get

$$I_2^{(d)}(m_1^2,m_2^2,p_{12}) = \frac{p_{12}+m_1^2-m_2^2-\alpha_{12}}{2p_{12}}\, I_2^{(d)}(m_1^2,0,s_{13}) + \frac{p_{12}-m_1^2+m_2^2+\alpha_{12}}{2p_{12}}\, I_2^{(d)}(0,m_2^2,s_{23})$$

where

$$s_{13}=\frac{\Delta_{12}+2p_{12}m_1^2-(p_{12}+m_1^2-m_2^2)\alpha_{12}}{2p_{12}},$$

$$s_{23}=\frac{\Delta_{12}+2p_{12}m_2^2+(p_{12}-m_1^2+m_2^2)\alpha_{12}}{2p_{12}},$$

$$\alpha_{12}=\pm\sqrt{\Delta_{12}}.$$

$$\Delta_{ij}=p_{ij}^2+m_i^4+m_j^4-2p_{ij}m_i^2-2p_{ij}m_j^2-2m_i^2m_j^2.$$

Integral with arbitrary masses and momentum can be expressed in terms of integrals with one propagator massless !!!

One-loop vertex type integral

At $n=4,\ j=1,\ m_4=0$, imposing conditions $G_3=0,\ \Delta_4=0$ we get

$$
\begin{aligned}
& I_3^{(d)}(m_1^2, m_2^2, m_3^2, s_{23}, s_{13}, s_{12}) = \\
& \frac{s_{13}+m_3^2-m_1^2+\alpha_{13}}{2s_{13}} \\
& \times I_3^{(d)}(m_2^2, m_3^2, 0, s_{34}^{(13)}, s_{24}(m_1^2, m_3^2, s_{23}, s_{13}, s_{12}), s_{23}) \\
& + \frac{s_{13}-m_3^2+m_1^2-\alpha_{13}}{2s_{13}} \\
& \times I_3^{(d)}(m_1^2, m_2^2, 0, s_{24}(m_1^2, m_3^2, s_{23}, s_{13}, s_{12}), s_{14}^{(13)}, s_{12})
\end{aligned}
$$

Again as it was for integral $I_2^{(d)}$ integral $I_3^{(d)}$ with arbitrary arguments can be expressed in terms of integrals with at least one propagator massless!!!

Derivation of FE from dimensional invariance

It will be highly desirable to derive functional equations from some invariance principle.

Feynman integrals depend on two rather different sets of variables: kinematical variables and masses as well as on parameter of the space time dimension d and powers of propagators.

The general form of functional equations for a given integral $I^{(d)}(\{x_j\}$:

$$\sum_j x_j(d, \{s_{ij}\}, \{m_k\})\ I^{(d)}(\{s_{ij}\}, \{m_k^2\}) = 0.$$

where x_j, m_j are different sets of kinematical variables, and d is arbitrary.
Functional equations must be valid for arbitrary value of d.
We can take this property as a basic principle for finding functional equations among Feynman integrals.

Namely, we will try to find some combination of integrals in the form

$$\Phi(d) = \sum_j x_j(d, \{s_{ij}\}, \{m\}) \; I^{(d)}(\{s_{ij}\}, \{m_j^2\}).$$

which is invariant with respect to shift $d \to d+2$, i.e.

$$\Phi(d+2) = \Phi(d),$$

Any integral with space time dimension $d+2$ can be expressed in terms finite set of basis integrals with space time dimension d (O.V.T, Phys.Rev. D54 (1996) 6479).
This dimensional recurrence relations can be used for constructing dimensionally invariant combinations of integrals with different arguments and discovering functional equations.

Algorithm for deriving functional equations

Algorithm for deriving functional equations :

- For a given integral write combination $\Phi(d)$ of this integral with undetermined arguments $\{s_{ij}\}$, m_k^2 and undetermined coefficients x_k
- Shift space time dimension by 2: $d \to d+2$ and express integrals $I^{(d+2)}$ in terms of integrals with dimension d .
- By equating arguments of integrals and coefficients in front of them derive equations for undetermined arguments and coefficients.
- Find additional equations for x_j by considering limiting cases like $|d| \to \infty$, $d \to d_0$ $(d_0 = 0, 1, 2, 3...)$ or by taking imaginary part of integrals on some cut.
- Check that the discovered dimensionally invariant combination is zero by considering particular cases $s_{ij} = s_{ij}^{(0)}$ and/or by differentiating it with respect to kinematical variables and masses.

One-loop example

Let's consider simple example:

$$I_2^{(d)}(q^2,m^2) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{(k_1^2+i\eta)((k_1-q)^2-m^2+i\eta)}$$

According to our proposal we write:

$$\Phi(d) = I_2^{(d)}(q^2,m^2) + x_1 I_2^{(d)}(q_1^2,m_1^2) + x_2 I_2^{(d)}(q_2^2,m_2^2) = 0.$$

We can try to find functional equation with x_j independent of d. Then

$$\Phi(d+2) = I_2^{(d+2)}(q^2,m^2) + x_1 I_2^{(d+2)}(q_1^2,m_1^2) + x_2 I_2^{(d+2)}(q_2^2,m_2^2) = 0.$$

One-loop propagator integral

Integral $I_2^{(d)}(q^2, m^2)$ satisfies the following dimensional recurrence relation:

$$I_2^{(d+2)}(q^2, m^2) = \frac{(m^2 - q^2)^2}{2q^2(d-1)} I_2^{(d)}(q^2, m^2) - \frac{(m^2 + q^2)}{2q^2(d-1)} T_1^{(d)}(m^2),$$

where

$$T_1^{(d)}(m^2) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{k_1^2 - m^2 + i\eta} = -\Gamma\left(1 - \frac{d}{2}\right) m^{d-2}$$

Substituting this relation into $\Phi(d+2)$ gives:

$$z_0 I_2^{(d)}(q^2, m^2) + z_1 I_2^{(d)}(q_1^2, m_1^2) + z_2 I_2^{(d)}(q_2^2, m_2^2) + z_3 T_1^{(d)}(m^2) + z_4 T_1^{(d)}(m_1^2) + z_5 T_1^{(d)}(m_2^2) = 0.$$

One-loop propagator integral

Solving equation $\Phi(d) = 0$ with respect to $I_2^{(d)}(q^2, m^2)$ and substituting solution into the new equation leads to the following relation:

$$Y_1 I_2^{(d)}(q_1^2, m_1^2) + Y_2 I_2^{(d)}(q_2^2, m_2^2) - \frac{(m^2+q^2)}{2q^2(d-1)} T_1^{(d)}(m^2)$$
$$-\frac{(m_1^2+q_1^2)}{2(d-1)q_1^2} x_1 T_1^{(d)}(m_1^2) - \frac{(m_2^2+q_2^2)}{2(d-1)q_2^2} x_2 T_1^{(d)}(m_2^2) = 0,$$

where

$$Y_1 = -\frac{(-q^2 m_1^4 + 2q^2 m_1^2 q_1^2 - q^2(q_1^2)^2 + q_1^2 m^4 - 2q_1^2 m^2 q^2 + q_1^2(q^2)^2)}{2(d-1)q^2 q_1^2} x_1,$$

$$Y_2 = -\frac{(q_2^2 m^4 - 2q_2^2 m^2 q^2 + q_2^2(q^2)^2 - q^2 m_2^4 + 2q^2 m_2^2 q_2^2 - q^2(q_2^2)^2)}{(d-1)q^2 q_2^2} x_2.$$

One-loop propagator integral

We assume that the sets $\{q_1^2, m_1^2\}$, $\{q_2^2, m_2^2\}$ are different and integrals depending on these sets are independent and nontrivial. Therefore it must be

$$Y_1 = 0, \qquad\qquad Y_2 = 0.$$

These equations can be solved for q_1^2 and q_2^2 After that three terms will remain

$$-\frac{(m^2+q^2)}{2q^2(d-1)} T_1^{(d)}(m^2)$$
$$-\frac{(m_1^2+q_1^2)}{2(d-1)q_1^2} x_1 T_1^{(d)}(m_1^2) - \frac{(m_2^2+q_2^2)}{2(d-1)q_2^2} x_2 T_1^{(d)}(m_2^2) = 0,$$

In order to cancel the term with $T_1^{(d)}(m^2)$ and get rid off all $T_1^{(d)}$ terms one should assume either

$$m_1^2 = m^2,\ m_2^2 = 0, \quad \text{or} \quad m_1^2 = 0, \quad m_2^2 = m^2, \quad \text{or} \quad m_1^2 = m_2^2 = m^2.$$

One-loop propagator integral

At $m_1^2 = m^2, \; m_2^2 = 0$

$$x_1 = -\frac{m^2}{q^2}, \quad q_1^2 = \frac{m^4}{q^2}, \quad q_2^2 = \frac{(q^2 - m^2)^2}{q^2}$$

At $m_1^2 = 0, \; m_2^2 = m^2$

$$x_2 = -\frac{m^2}{q^2}, \quad q_1^2 = \frac{(q^2 - m^2)^2}{q^2}, \quad q_2^2 = \frac{m^4}{q^2}$$

In order to determine both x_1 and x_2 for both solutions additional equation is required. It can be easily found by taking the limit $(4 - d)/2 = \varepsilon \to 0$ for $\Phi(d)$:

$$1 + x_1 + x_2 = 0.$$

One-loop propagator integral

At $m_1^2=m^2,\ m_2^2=m^2$ as follows from equations $Y_1=Y_2=0$:

$$q_1^2=q_2^2=\frac{m^4}{q^2},$$

that from one side contradicts to our assumption about different sets of arguments and from the other side it leads to the dimensional invariant

$$\Phi(d)\neq 0.$$

In both considered cases we obtained the same functional equation:

$$I_2^{(d)}(q^2,m^2)=\frac{m^2}{q^2}I_2^{(d)}\left(\frac{m^4}{q^2},m^2\right)+\frac{(q^2-m^2)}{q^2}I_2^{(d)}\left(\frac{(q^2-m^2)^2}{q^2},0\right).$$

where

$$I_2^{(d)}(q^2,0)=\frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma^2\left(\frac{d}{2}-1\right)}{\Gamma(d-2)}(-q^2)^{\frac{d}{2}-2}.$$

One-loop box type integral

The same algorithm can be applied to more complicated integrals. We considered the box-type integral $I_4^{(d)}(\{s_{kr}\})$

$$I_4^{(d)}(s_{nj}, s_{jk}, s_{kl}, s_{nl};\ s_{jl}, s_{nk}) = \int \frac{d^d q}{i\pi^{d/2}}$$
$$\times \frac{1}{[(q-p_n)^2 - m^2][(q-p_j)^2 - m^2][(q-p_k)^2 - m^2][(q-p_l)^2 - m^2]},$$

where

$$s_{ij} = (p_i - p_j)^2, \qquad m^2 = \mu^2 - i\eta.$$

taken at $s_{12} = s_{23} = s_{34} = 0$.

One-loop box type integral

We derived a new functional equation for the integral with this kinematics:

$$I_4^{(d)}(s_{14}, s_{24}, s_{13}, m^2) = \frac{s_{13}}{s_{13}-s_{14}} \; I_4^{(d)}\left(0, \frac{s_{13}s_{24}}{s_{13}-s_{14}}, s_{13}, m^2\right)$$
$$-\frac{s_{14}}{s_{13}-s_{14}} \; I_4^{(d)}\left(s_{24}, \frac{s_{13}s_{24}}{s_{13}-s_{14}}, s_{14}, m^2\right),$$

where

$$I_4(s_{14}, s_{24}, s_{13}, m^2) \equiv I_4(0,0,0,s_{14}, s_{24}, s_{13}, m^2).$$

The most complicated part in derivation of this equation was analysis of all possible solutions of equations for unknown parameters and arguments of integrals. The procedure of finding all possible solutions was implemented in Maple and also in Mathematica. Sometimes Maple discovered not all solutions.

Question

We used recurrence relations to derive FE. All recurrence relations follow from the equation:

$$\int d^d k \frac{\partial}{\partial k_\mu} f(k, \{s_{ij}\}, \{m_r^2\}) = 0.$$

One can raise the question:
Functional equations hold for integrals or they can be obtained as a consequence of a relation between integrands?

Algebraic relations for propagators

Analyzing one-loop FE one can see that integrands are rather similar and differ only by one propagator:

Integrands for the one-loop propagator type integrals

$$\frac{1}{D_1 D_2}, \quad \frac{1}{D_0 D_2}, \frac{1}{D_1 D_0},$$

Integrands for the one-loop vertex type integrals

$$\frac{1}{D_1 D_2 D_3}, \quad \frac{1}{D_0 D_2 D_3}, \frac{1}{D_1 D_0 D_3}, \frac{1}{D_1 D_2 D_0}$$

where

$$D_0 = (k_1 - p_0)^2 - m_0^2 + i\eta, \quad D_1 = (k_1 - p_1)^2 - m_1^2 + i\eta,$$
$$D_2 = (k_1 - p_2)^2 - m_2^2 + i\eta, \quad D_3 = (k_1 - p_3)^2 - m_3^2 + i\eta,$$

Observation: since $G_n = 0$ vectors $p_1, p_2, \ldots$ are linearly dependent

Algebraic relations for propagators

Question: Would it be possible to find algebraic relations of the form:

$$\frac{1}{D_1 D_2} = \frac{x_1}{D_0 D_2} + \frac{x_2}{D_1 D_0}$$

where

$$p_0 = y_{01} p_1 + y_{02} p_2$$

and $x_1, x_2, \; y_{01}, y_{02}$ being independent of k_1.

The answer is - YES! Putting all terms over the common denominator and equating coefficients in front of different products of $(k_1^2)^a (k_1 p_1)^b (k_1 p_2)^c$ to zero we obtain system of equations:

$$y_{02} - x_2 = 0, \quad y_{01} - x_1 = 0, \quad x_1 + x_2 = 1,$$
$$p_1^2 (x_1 - y_{01}^2) + p_2^2 (x_2 - y_{02}^2) + y_{01} y_{02} (s_{12} - p_1^2 - p_2^2) - m_1^2 x_1 - m_2^2 x_2 + m_0^2 = 0.$$

where $s_{12} = (p_1 - p_2)^2$

Algebraic relations for propagators

Solution of this system of equations is:

$$x_1 = y_{01} = \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} - \frac{\sqrt{\Lambda_2 + 4s_{12}m_0^2}}{2s_{12}},$$

$$x_2 = y_{02} = \frac{m_1^2 - m_2^2 + s_{12}}{2s_{12}} + \frac{\sqrt{\Lambda_2 + 4s_{12}m_0^2}}{2s_{12}}.$$

and

$$\Lambda_2 = s_{12}^2 + m_1^4 + m_2^4 - 2s_{12}(m_1^2 + m_2^2) - 2m_1^2 m_2^2.$$

Algebraic relations for propagators

Integrating obtained algebraic relation w.r.t. k_1 gives the following FE:

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = \frac{s_{12} + m_1^2 - m_2^2 - \lambda}{2s_{12}} \; I_2^{(d)}(m_1^2, m_0^2, s_{13}(m_1^2, m_2^2, m_0^2, s_{12})) + \frac{s_{12} - m_1^2 + m_2^2 + \lambda}{2s_{12}} \; I_2^{(d)}(m_2^2, m_0^2, s_{23}(m_1^2, m_2^2, m_0^2, s_{12})).$$

where

$$s_{13} = \frac{\Lambda_2 + 2s_{12}(m_1^2 + m_0^2)}{2s_{12}} + \frac{m_1^2 - m_2^2 + s_{12}}{2s_{12}}\lambda$$

$$s_{23} = \frac{\Lambda_2 + 2s_{12}(m_2^2 + m_0^2)}{2s_{12}} + \frac{m_1^2 - m_2^2 - s_{12}}{2s_{12}}\lambda.$$

$$\lambda = \sqrt{\Lambda_2 + 4s_{12}m_0^2}$$

Parameter m_0 is arbitrary and can be taken at will. The same equation was obtained from recurrence relations by imposing conditions on Gram determinants.

Algebraic relations for propagators

Similar to the relation with two propagators one can find relation for three propagartors:

$$\frac{1}{D_1 D_2 D_3} = \frac{x_1}{D_2 D_3 D_0} + \frac{x_2}{D_1 D_3 D_0} + \frac{x_3}{D_1 D_2 D_0}.$$

Here p_1, p_2 and p_3 are independent external momenta, k_1 will be integration momentum and

$$p_0 = y_{01} p_1 + y_{02} p_2 + y_{03} p_3.$$

Multiplying both sides of equation by the product $D_1 D_2 D_3 D_0$ and equating coefficients in front of $k_1^2, k_1 p_1, k_1 p_2, k_1 p_3$ and term independent of k_1 we obtain system of equations

$$y_{01} - x_1 = 0, \quad y_{02} - x_2 = 0, \quad y_{03} - x_3 = 0, \quad x_3 + x_2 + x_1 - 1 = 0,$$
$$[x_1 - y_{01}(y_{03} + y_{02} + y_{01})]p_1^2 + [x_2 - y_{02}(y_{03} + y_{02} + y_{01})]p_2^2$$
$$+[x_3 - y_{03}(y_{03} + y_{01} + y_{02})]p_3^2$$
$$+y_{02} y_{03} p_{23} + y_{01} y_{03} p_{13} + y_{01} y_{02} p_{12} - m_1^2 x_1 - m_2^2 x_2 - m_3^2 x_3 + m_0^2 = 0.$$

Algebraic relations for product of 3 propagators

This system has the following solution

$$x_1 = y_{01} = 1 - \alpha - y_{02}, \qquad x_2 = y_{02}, \qquad x_3 = y_{03} = \alpha,$$

where α is solution of the quadratic equation

$$\alpha^2 p_{13} + [m_3^2 - m_1^2 - p_{13} + y_{02}(p_{13} + p_{12} - p_{23})]\alpha$$
$$+m_1^2 - m_0^2 + (m_2^2 - m_1^2 - p_{12} + p_{12} y_{02}) y_{02} = 0.$$

Solution depends on 2 arbitrary parameters: $m_0,\ y_{02}$.
By integrating the obtained relation we get the same FE as it was given before.
Functional relations for Feynman integrals with integrands being rational functions strongly remind *Abel's addition theorem!*

Abelian integral is an integral in the complex plane of the form

$$\int_{z_0}^{z} R(x,y)dx,$$

where $R(x,y)$ is an arbitrary *rational* function of the two variables x and y. These variables are related by the equation

$$F(x,y) = 0,$$

where $F(x,y)$ is an irreducible polynomial in y,

$$F(x,y) \equiv \phi_n(x)y^n + ... + \phi_1(x)y + \phi_0(x),$$

whose coefficients $\phi_j(x)$, $j = 0, 1, ...n$ are rational functions of x. Abelian integrals are natural generalizations of elliptic integrals, which arise when

$$F(x,y) = y^2 - P(x),$$

where $P(x)$ is a polynomial of degree 3 and 4. If degree of the polynomial is greater than 4 then we have *hyperelliptic integral*.

Abel's theorem

Let C and C' be plane curves given by the equations

$$\begin{aligned} C: \quad & F(x,y) = 0, \\ C': \quad & \phi(x,y) = 0. \end{aligned}$$

These curves have n points of intersections $(x_1, y_1), \ldots (x_n, y_n)$, where n is the product of degrees of C and C'. Let $R(x, y)$ be a rational function of x and y where y is defined as a function of x by the relation $F(x, y) = 0$. Consider the sum

$$I = \sum_{i=1}^{n} \int_{x_0, y_0}^{x_i, y_i} R(x,y) dx$$

Integrals being taken from a fixed point to the n points of intersections. If some of the coefficients $a_1, a_2, \ldots, a_k$ of $\phi(x, y)$ are regarded as continuous variables, the points (x_i, y_i) will vary continuously and hence I will be a function, whose form is to be determined, of the variable coefficients $a_1, a_2, \ldots, a_k$.

Abel's theorem

Abel's theorem:
The partial derivatives of the sum I , with respect to any of the coefficients of the variable curve $\phi(x, y) = 0$, is a *rational* function of the coefficients and hence I is equal to a *rational* function of the coefficients of $\phi(x, y) = 0$, plus a finite number of logarithms or arc tangents of such rational functions.
Important: integrals themselves can be rather complicated transcendental functions but their sum can be simple.
Example: Elliptic integral of the second type:

$$E(k, x) = \int_0^x \frac{(1 - k^2 x)dx}{\sqrt{x(1 - x)(1 - k^2 x)}}.$$

Abel's theorem

We take as C and C'

$$\begin{aligned} C: \quad & y^2 = x(1-x)(1-k^2x), \\ C': \quad & y = ax + b. \end{aligned}$$

The elimination of y between two equations will give us as the abscissae x_1, x_2, x_3 of the points of intersection the three roots of the equation:

$$\phi(x) = k^2x^3 - (1 + k^2 + a^2)x^2 + (1 - 2ab)x - b^2 = 0.$$

The corresponding sum will be

$$I(a,b) = \int_0^{x_1} R(x)dx + \int_0^{x_2} R(x)dx + \int_{1/k^2}^{x_3} R(x)dx$$

Abel's theorem gives addition formula:

$$\int_0^{x_1} R(x)dx + \int_0^{x_2} R(x)dx + \int_{1/k^2}^{x_3} R(x)dx = -2a + \kappa,$$

where κ is an arbitrary constant.

One can find FE for Feynman integrals following closely derivation of relationships for usual algebraic integrals.
Deriving relations for propagators we used orthogonality condition

$$G_n = 0. \tag{1}$$

In fact it is not needed to assume such a relation. For example, to fix parameters in algebraic relations for products of two propagators

$$R_2(k_1, p_1, p_2, m_1^2, m_2^2, m_0^2) = \frac{1}{D_1 D_2} - \frac{x_1}{D_2 D_0} - \frac{x_2}{D_1 D_0} = 0, \tag{2}$$

instead of (1) we can impose conditions

$$\frac{\partial x_1}{\partial k_{1\mu}} = \frac{\partial x_2}{\partial k_{1\mu}} = 0. \tag{3}$$

Multiplying both sides of (2) by $D_0 D_1 D_2$ we get

$$\begin{aligned} &D_0 - x_1 D_1 - x_2 D_2 \\ &= (1 - x_1 - x_2) k_1^2 + 2x_1 k_1 p_1 + 2x_2 k_1 p_2 \\ &+ x_1 m_1^2 + x_2 m_2^2 - x_1 p_1^2 - x_2 p_2^2 - 2k_1 p_3 - m_3^2 + p_3^2. \end{aligned} \tag{4}$$

Differentiating this relation, contracting with k_1, p_1, p_2, p_3 and taking into account (3) gives several equations:

$$
\begin{aligned}
&-2x_1(k_1^2 - k_1p_1) - 2x_2(k_1^2 - k_1p_2) + 2k_1^2 - 2k_1p_3 = 0,\\
&2(1 - x_1 - x_2)k_1p_1 + 2x_1p_1^2 + 2x_2p_1p_2 - 2p_1p_3 = 0,\\
&2(1 - x_1 - x_2)k_1p_2 + 2p_1p_2x_1 + 2x_2p_2^2 - 2p_2p_3 = 0,\\
&2(1 - x_1 - x_2)k_1p_3 + 2x_1p_1p_3 + 2x_2p_2p_3 - 2p_3^2 = 0.
\end{aligned}
\tag{5}
$$

They can be used to express $k_1p_3,\ p_1p_3,\ p_2p_3,\ x_1,\ x_2$ in terms of $k_1^2,\ k_1p_1,\ k_1p_2,\ p_1^2,\ p_1p_2,\ p_2^2$ considered to be independent variables.
For example, we get:

$$
\begin{aligned}
&k_1p_3 = x_1k_1p_1 - x_1k_1p_2 + k_1p_2 + \frac{x_1}{2}(m_1^2 - m_2^2 - p_1^2 + p_2^2)\\
&+\frac{1}{2}(m_2^2 - p_2^2 - m_3^2 + p_3^2),
\end{aligned}
$$

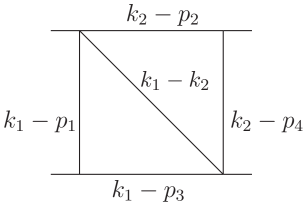
and similar expressions for other scalar products of p_3. Solution for $x_1,\ x_2$ is the same as it was before and as a result we obtained the same relation between products of two propagators.

- Solution of the above system of equations is rather similar to finding intersections of two plane curves considered in Abel's theorem.
- Similar to usual algebraic integrals for one variable we can construct various integrands out of our different relationships for products of propagator. These integrands will be rational functions in independent variables.
- Integrations should be done over d dimensional space. Rational function must resemble integrands for Feynman integrands.

For example, integrating product of different relationships between two propagators multiplied by $1/[(k_1 - k_2)^2 - m_5^2]^{\nu_5}$ with respect to k_1, k_2 leads to the FE for the integral

$$\int \frac{d^d k_1 d^d k_2}{[(k_1 - k_2)^2 - m_5^2]^{\nu_5}} R_2(k_1, p_1, p_2, m_1^2, m_2^2, m_0^2) R_2(k_2, p_3, p_4, m_3^2, m_4^2, \widetilde{m}_0^2) = 0$$

corresponding to the following diagram



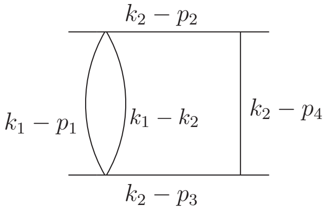
By integrating product of the relationship

$$R_3(k_1, p_1, p_2, p_3, m_1^2, m_2^2, m_3^2, m_0^2) = \frac{1}{D_1 D_2 D_3} - \frac{x_1}{D_2 D_3 D_0} - \frac{x_2}{D_1 D_3 D_0} - \frac{x_3}{D_1 D_2 D_0} = 0,$$

and one loop-propagator integral

$$\int\int \frac{d^d k_1 d^d k_2}{[k_1^2 - m_1^2]^{\nu_1}[(k_1-k_2)^2 - m_5^2]^{\nu_5}} R_3(k_2, p_2, p_3, p_4, m_2^2, m_3^2, m_4^2) = 0,$$

we obtain the FE for the integral corresponding to the following diagram with arbitrary ν_1, ν_5, and arbitrary momenta and masses



Concluding remarks

- Three different methods for finding FE for Feynman integrals with any number of loops and external legs are proposed.
- FE reduce integrals with complicated kinematics to simpler integrals
- FE can be used for analytic continuation of Feynman integrals without knowing explicit analytic result.
- Application of these methods for some two- and three- loop integrals is in progress
- Systematic investigation of FE for Feynman integrals based on algebraic geometry and group theory is needed.
- Some improvements of these methods can be done by exploiting known methods for algebraic integrals.
- The methods can be extended for finding functional equations among hypergeometric as well as holonomic functions.