GRACE Note
July 2002

# Kinematics Library in GRACE

Y. Kurihara

*High Energy Accelerator Research Organization,*
*Oho 1-1, Tsukuba, Ibaraki 305-0801, Japan*

# 1 Kinematics

## 1.1 Cross section formula

We use the normalization of a state in which the volume element for each of final particle is given by

$$\int \frac{\mathrm{d}^3 p}{2E(2\pi)^3}, \tag{1}$$

where $\mathbf{p}$ is the three momentum of the particle and $E$ is its energy.

In the case of a collision of two particles labeled by 1 and 2, the flux is given by

$$\text{flux} = v_{rel} 2E_1 2E_2, \quad v_{rel} = \left| \frac{\mathbf{p}_1}{E_1} - \frac{\mathbf{p}_2}{E_2} \right|, \tag{2}$$

with $v_{rel}$ being the relative velocity of particles 1 and 2. Hence the total cross section is written as

$$\sigma = \frac{1}{\text{flux}} \prod_j \int \frac{\mathrm{d}^3 p_j}{2E_j(2\pi)^3} (2\pi)^4 \delta^4(p_1 + p_2 - \sum_j p_j) \sum_{h_f} \sum_{h_i} |T_{if}|^2, \tag{3}$$

where $T_{if}$ is the total helicity amplitude for the process $i \to f$. Initial helicities are averaged and the final ones are summed. This is the formula of the cross section used in the system. In the built-in kinematics the cross section is given in the unit of *pico-barn*(pb).

For the decay of a particle with momentum $p$, the total decay rate is given by

$$\Gamma = \frac{1}{2E} \prod_j \int \frac{\mathrm{d}^3 p_j}{2E_j(2\pi)^3} (2\pi)^4 \delta^4(p - \sum_j p_j) \sum_{h_f} \sum_{h_i} |T_{if}|^2, \tag{4}$$

The independent integration variables, usually expressed by momenta and angles, must be mapped to those of the integration package `BASES` which are usually normalized as $0 < \mathtt{x(i)} < 1$ where `i` runs from 1 to `ndim`, the dimension of the integration,

## 1.2 Identical particles

One should remember that the statistical factor for the identical particles is *not* generated automatically and should be supplied by hand. This is because this factor depends on the observables one is interested. For the total cross section one should divide the result obtained by `BASES` by the factorial of the number of identical particles.

### 1.3 Kinematics database

We list the built-in kinematics in the system. This will be extended time to time to cover more cases. The title of the subsection below indicates the code number of the kinematics. For a scattering the total energy of the system should be given in `kinit.f`. The 4-vector of a momentum is defined as $(p_x, p_y, p_z, E)$ ordering, that is, in the program it is given by an array `pe(1,n),pe(2,n),pe(3,n),pe(4,n)` for the `n`-the particle.

In GRACE the kinematics assumes the following two processes:

$$p_1 + p_2 \to p_3 + p_4 + \cdots \qquad \text{(2-body scattering)},$$
$$p_1 \to p_2 + p_3 + p_4 + \cdots \qquad \text{(decay of a particle)}.$$

It should be remembered that the assignment of particles follows the order of particles originally defined in `in.prc`.

The Lorentz invariant phase space element for a final $n$-body $(A \to 1 + 2 + 3 + \cdots + n)$ is defined by

$$\begin{aligned} d\tilde{\Gamma}_n &= (2\pi)^4 \delta^4 \left( \sum_{in} p - \sum_{out}^{n} p \right) \prod_{out}^{n} \frac{d^3 p_j}{(2\pi)^3 2E_j} \equiv \frac{1}{(2\pi)^{3n-4}} d\Gamma_n, \\ d\Gamma_n &= \delta^4 \left( \sum_{in} p - \sum_{out}^{n} p \right) \prod_{out}^{n} \frac{d^3 p_j}{2E_j}. \end{aligned}$$

The following chain relation is useful$(0 < k < n-1)$:

$$d\Gamma_n(A \to 1 + 2 + 3 + \cdots) = d\Gamma_{k+1}(A \to 1 + \cdots + k + q) \frac{dQ^2}{2\pi} d\Gamma_{n-k}(q \to (k+1) + (k+2) + \cdots)$$

where $q^2 = Q^2$.

When $p_a$ is in the center-of-mass system,i.e., $\mathbf{p}_a = \mathbf{0}$, and if $p_a = p_b + p_c$ we introduce the 2-body phase space

$$d\Gamma_{\rm CM}(a; bc) \equiv d\Gamma_2 = \delta^4(p_a - p_b - p_c) \frac{d^3 p_b}{2E_b} \frac{d^3 p_c}{2E_c}$$

and write it by angular variables:

$$d\Gamma_{\rm CM}(a; bc) = \frac{\beta(a; bc)}{8} d\Omega_{\rm CM}(a; bc) = \frac{\beta(a; bc)}{8} d\cos\theta_{b,(bc)} d\phi_{b,(bc)}.$$

Here $\beta$ is given by

$$\begin{aligned} \beta(a; bc) &= \frac{2P}{E_a} \\ &= \frac{1}{E_a^2} \sqrt{(E_a + m_b + m_c)(E_a - m_b - m_c)(E_a + m_b - m_c)(E_a - m_b + m_c)} \end{aligned}$$

and the subscript $(bc)$ indicates that the angles $\theta_b, \phi_b$ are defined in the center-of-mass system. Angles in the laboratory frame have no subscript.

## 2 kinematics database

### 2.1 List of built-in kinematics

Here is a list of the built-in kinematics in the GRACE system. A code number of each kinematics is set to be (code number)$=I \times 1000 + J \times 100 + K$, where $I$ $(J)$ is a number of initial (final) state particles, and $K$ is a serial number of routines. Singularities treated in each kinematics are summarized in the table, and detailed descriptions can be found in following subsections.

| code number | contents |
|---|---|
| 1201 | 1-body $\to$ 2 body decay |
| 1301 | 1-body $\to$ 3 body decay<br>Sequential decay $1 \to 2 + (3+4) \to 2+3+4$ can be treated. |
| 2201 | 2-body $\to$ 2 body in CM frame<br>$t$- and $u$-channel singularities can be treated. |
| 2301 | 2-body $\to$ 3 body in CM frame ,<br>Sequential decay type $1+2 \to 3 + (4+5) \to 3+4+5$.<br>Resonance on particles 4 and 5 can be treated. |
| 2302 | 2-body $\to$ 3 body in CM frame ,<br>Radiative processes $1+2 \to 3(\gamma) + 4 + 5$,<br>both initial and final radiation can be treated. |
| 2303 | 2-body $\to$ 3 body in CM frame ,<br>Double-radiative processes $1+2 \to 3(\gamma) + 4(\gamma) + 5$ |
| 2304 | 2-body $\to$ 3 body in CM frame ,<br>Three photon processes $1+2 \to 3(\gamma) + 4(\gamma) + 5(\gamma)$ |
| 2401 | 2-body $\to$ 4 body in CM frame, a pair of sequential<br>decay type $1+2 \to (3+4) + (5+6) \to 3+4+5+6$<br>$t$-channel singularity can be treated. |
| 2402 | 2-body $\to$ 4 body in CM frame,<br>'fusion' type $1+2 \to (3+A) + (4+B); A+B \to 5+6$ |

### 2.2 Input/Output variables

Each kinematics code consists of two subroutines, `kinem.f` and `kinit.f`. A `kinit.f` treats initialization of parameters used in the kinematics, and a `kinem.f` is kinematics itself. `kinit.f` has eight arguments as:

- `NEXTRN`(input): a number of external particles given by GRACE,
- `X(Ndim)`(input): random numbers given by `BASES`,
- `PE(4,NEXTRN)`(output): four vecters of external particles,
- `PP(NEXTRN,NEXTRN)`(output): invariants between two external particles,
- `YACOB`(output): Jacobian for a given phase point including a flux factor and a conversion factor from (GeV) -2 to $pb$ (for the scattering kinematics),
- `NREG`(input): a number of sub-region in the phase space,
- `IREG`(input): a pointer of the sub-region,
- `JUMP`(output): a flag showing this point is acceptable (`JUMP=0`) or not (`JUMP=1`).

### 2.3 1201

This is a simple two-body decay kinematics.

$$d\tilde{\Gamma}_2 = \frac{1}{(2\pi)^2} d\Gamma_{\rm CM}(1;23) = \frac{\beta(1;2,3)}{8\pi}.$$

Here $\beta$ is given by

$$\beta(a;b,c) = \frac{2P_b}{M_a}.$$

**Options in `kinit.f`**

1. Physical parameters
   *non* (Particle masses are given by GRACE through *amass1(I)*.)

2. Control parameters
   *non*

### 2.4 1301

This is the kinematics for 3 body decay.

$$d\tilde{\Gamma}_3 = \frac{1}{(2\pi)^5} d\Gamma_{\rm CM}(1;234) = \frac{1}{(2\pi)^3}\frac{1}{16M_1^2} dq_{23}^2 dq_{34}^2,$$

where $q_{ij}^2 = (p_i + p_j)^2$. A resonance production in particles 2 and 3 can be treated in this kinematics.

**Options in `kinit.f`**

1. Physical parameters

   - `ARESNS(1)` : Mass of a resonance in particles 2,3.
   - `ARESNS(2)` : Width of a resonance in particles 2,3.

2. Control parameters

   - `IRESNS` : Treatment of $q_{23}^2$.
     `= 0` : no-singularity
     `= 1` : narrow resonance (single)

### 2.5 2201

This is the kinematics for 2 to 2 process in the center-of mass system. Integration variables are naturally the polar angle $\theta$ and azimuthal angle $\phi$ with respect to the incoming particles.

Phase space is in the center-of-mass system and it is given by

$$d\tilde{\Gamma}_2 = \frac{1}{(2\pi)^2} d\Gamma_{\rm CM}(12;34) = \frac{\beta(1+2;3,4)}{8(2\pi)^2} d\cos\theta d\phi.$$

**Options in `kinit.f`**

1. Physical parameters

   - `W` : Center of mass energy.
   - `COSCUT(1)` : Minimum of $\cos\theta$.
   - `COSCUT(2)` : Maximum of $\cos\theta$.

2. Control parameters

   - `ICOST` :Collinear singularity treatment.
     `= 0` : No singularity.
     `= +1`: $\sim 1/t$ singularity, where $t = (p1 - p3)^2$.
     `= -1`: $\sim 1/t$ and $\sim 1/u$ singularity, where $u = (p1 - p4)^2$.

## 2.6 2301

This is the kinematics for 2 to 3 process in the center-of mass system. Here the final state first splits into particle-3 and the system of particles 4 and 5. After that the latter decays into particle-4 and particle-5.

$$1 + 2 \longrightarrow 3 + q \qquad q \longrightarrow 4 + 5$$

Integration variables are the polar angle $\theta$ and azimuthal angle $\phi$ for the first split, those angles for the second split, and the invariant mass of particles 4 and 5.

Phase space is in the center-of-mass system and it is given by

$$\begin{aligned} d\tilde{\Gamma}_3 &= \frac{1}{2\pi} d\Gamma_2(1+2 \to 3+q) dQ^2 d\Gamma_2(q \to 4+5) \\ &= \frac{\beta(12;3q)\beta(q;45)}{8^2(2\pi)^5} d\cos\theta_3 d\phi_3 dQ^2 d\cos\theta_{4,(45)} d\phi_{4,(45)}. \end{aligned}$$

The angles $\theta_3$ and $\phi_3$ represent the direction of particle-3. Invariant mass of 4 and 5, $Q^2$, is another variable.

$$Q^2 = (p_4 + p_5)^2$$

In the center of mass system of particles 4 and 5, angles $\theta_{4,(45)}$ and $\phi_{4,(45)}$ represent the direction of particle-4. The system of particles 4 and 5 are boosted backward to the momentum direction of particle-3.

This kinematics can treat processes in which particle 4 and 5 come from two independent resonances (for example, $b\bar{b}$ from a Higgs boson and a Z-boson). Moreover it can treat some singularities of angular distribution of particle-3 and particle-4.

With flag `ICOS3=1`, the cross section is assumed to have $\sim 1/t$ singularity, where $t = (p1 - p3)^2$. New integration variable is introduced as

$$\begin{aligned} D &= -t, \\ &= 2(E_1 E_3 - P_1 P_3 \cos\theta_3). \end{aligned}$$

$\cos\theta_3$ is measured with respect to $P_1$. The variable is transformed into

$$\begin{aligned} dD &= 2P_1 P_3 d\cos\theta, \\ dD/D &= d(\log D). \end{aligned}$$

Then the phase space in `2301` is replaced by

$$\begin{aligned} d\cos\theta_3 &= \frac{D}{2P_1P_3} d(\log D), \\ &= \frac{D}{2P_1P_3} \log(D_{max}/D_{min}) d\eta \end{aligned}$$

where $D = D_{min}(D_{max}/D_{min})^\eta$ for $0 < \eta < 1$. With flag `ICOS3=2`, $\cos\theta_3$ is symmetrizing around 90°.

With flag `ICOS4=1`, the cross section is assumed to have flat rapidity distribution. New integration variable $\eta$ is introduced as

$$-\log(1 + 2/\epsilon) < \eta < \log(1 + 2/\epsilon),$$
$$\epsilon = 2m_4^2/(P_4 + P_5)^2.$$

By using this variable, $\cos\theta_4$ can be expressed as;

$$\cos\theta_4 = (1 + \epsilon)\tanh\eta.$$

$\cos\theta_4$ is measured with respect to $-P_3$.

With flag `ICOS4=-1(-2)`, the cross section is assumed to have $\sim 1/t(+1/u)$ singularity, where $t = (p1 - p4)^2$ and $u = (p2 - p4)^2$. Similar treatment with `ICOS3=1 or 2` is done for $cos\theta_4$.

**Options in `kinit.f`**

1. Physical parameters

   - `W` : Center of mass energy.
   - `COSCUT(1,1)` : Minimum of $\cos\theta_3$.
   - `COSCUT(2,1)` : Maximum of $\cos\theta_3$.
   - `COSCUT(1,2)` : Minimum of $\cos\theta_4$. Here, $\theta_4$ is the polar angle of particle-4 with respect to the beam axis.
   - `COSCUT(2,2)` : Maximum of $\cos\theta_4$.
   - `COSCUT(1,3)` : Minimum of $\cos\theta_5$. Here, $\theta_5$ is the polar angle of particle-5 with respect to the beam axis.
   - `COSCUT(2,3)` : Maximum of $\cos\theta_5$.
   - `ENGYCT(1,1)` : Minimum of $E_3$.
   - `ENGYCT(2,1)` : Maximum of $E_3$.
   - `ENGYCT(1,2)` : Minimum of $E_4$.
   - `ENGYCT(2,2)` : Maximum of $E_4$.
   - `ENGYCT(1,3)` : Minimum of $E_5$.
   - `ENGYCT(2,3)` : Maximum of $E_5$.
   - `AMASCT(1)` : Minimum of $Q$. $Q$ is the mass of the system of particles 4 and 5.
   - `AMASCT(2)` : Maximum of $Q$.
   - `ARESNS(1,1)` : Mass of first resonance.
   - `ARESNS(2,1)` : Width of first resonance.
   - `ARESNS(1,2)` : Mass of second resonance.
   - `ARESNS(2,2)` : Width of second resonance.

2. Control parameters

   - `IRESNS` : Treatment of $Q^2$.
     `= 0` : no-singularity
     `= 1` : narrow resonance (single)
     `=-1` : narrow resonance (single)+1/Q2 singularity
     `= 2` : narrow resonance (double)
     `=-2` : narrow resonance (double)+1/Q2 singularity
     `= 3` : Q2=S peak
     `=-3` : 1/Q2 singularity only
     `= 4` : narrow resonance (single)+Q2=S peak

   - `ICOS3` : Treatment of $\theta_3$.
     `= 0` : no-singularity
     `= 1` : 1/t singularity
     `= 2` : 1/t + 1/u singularity

   - `ICOS4` : Treatment of $\theta_4$.
     `= 0` : no-singularity
     `= 1` : z-axis=-p3 ; flat rapidity
     `=-1` : z-axis=p1 ; 1/t singularity
     `=-2` : z-axis=p1 ; 1/t + 1/u singularity

## 2.7 2302

This is a kinematics for the radiative processes. A particle 3 is assumed to be a photon. To treat both initial and final state radiations, a phase space with respect to photon angles is divided into three regions;

$$
\begin{aligned}
\sigma &= \int_S \frac{d\sigma}{d\Omega_3} d\Omega_3, \\
&= \int_{S_{12}} \frac{d\sigma}{d\Omega_3} d\Omega_3 + \int_{S_4} \frac{d\sigma}{d\Omega_3} d\Omega_3 + \int_{S_5} \frac{d\sigma}{d\Omega_3} d\Omega_3, \\
S_{12} &= \{\hat{P}_3 | min\{\theta_{13}, \theta_{23}, \theta_{43}, \theta_{53}\} = \theta_{13} \ or \ \theta_{23}\}, \\
S_4 &= \{\hat{P}_3 | min\{\theta_{13}, \theta_{23}, \theta_{43}, \theta_{53}\} = \theta_{34}\}, \\
S_5 &= \{\hat{P}_3 | min\{\theta_{13}, \theta_{23}, \theta_{43}, \theta_{53}\} = \theta_{35}\},
\end{aligned}
$$

where $\theta_{ij} = cos^{-1}(\hat{P}_i \cdot \hat{P}_i)$ and $\hat{P}_i$ is a unit (three) vector along a three momentum of a particle-$i$.

**In the region $S_{12}$**

Phase space is the same as the `2301` except following modification to treat the collinear singularity: The energy of particle-3, $E_3$, is used instead of $Q^2$ using

$$Q^2 = W^2 - 2WE_3 + m_3^2 \quad (m_3 = 0)$$

and $E_3$ is converted into

$$\frac{dE_3}{E_3} = d(\log E_3)$$

to absorb the $1/E_3$ behavior(soft singularity) which appears in the photon radiation.

The angle $\theta_3$ is also changed to absorb the collinear singularity. It appears in the form of

$$\frac{1}{D_1 D_2}, \quad D_1 = 2p_1p_3, \quad D_2 = 2p_2p_3.$$

We introduce a variable

$$\tau = \frac{1 + v\cos\theta_3}{1 - v\cos\theta_3} \quad , \quad v = \sqrt{1 - 4m_1^2/W^2},$$

$$y = \frac{1}{4}\left(2 + \frac{\log\tau}{\log\xi}\right), \quad \xi = \sqrt{\frac{1+v}{1-v}}.$$

Here the correspondence is that $y = (0, 1)$ to $\cos\theta_3 = (-1, 1)$ and

$$D_1 = \frac{4E_1E_3}{1+\tau}, \quad D_2 = D_1\tau.$$

Then the phase space in `2301` is replaced by

$$dQ^2 = 2WE_3 \log(E_{3,max}/E_{3,min}) dx_4$$

where $E_3 = E_{3,min}(E_{3,max}/E_{3,min})^{x4}$ and

$$d\cos\theta_3 = \log(\xi) \frac{D_1D_2}{E_3^2} \frac{1}{2E_1P_1} dy.$$

**In the region $S_4$**

In the region $S_4$, the final state first splits into particle-5 and the system particle-4 and photon (particle-3). After that the latter decays into particle-4 and photon.

$$1 + 2 \longrightarrow 5 + q \quad , \quad q \longrightarrow 3(photon) + 4$$

Integration variables are the polar angle $\theta$ and azimuthal angle $\phi$ for the first split, those angles for the second split, and the invariant mass of particles 3 and 4. Angles for the first split are defined with respect to the incoming particles and those for the second split are defined in the center-of-mass system of particles 3 and 4 with respect to the momentum direction of the system.

The angles $\theta_4$ and $\phi_4$ represent the direction of particle-4. Invariant mass of 3 and 4, $Q^2$ is another variable.

$$Q^2 = (p_3 + p_4)^2$$

In the center of mass system of particles 3 and 4, angles $\theta_{3,(34)}$ and $\phi_{3,(34)}$ represent the direction of particle-3. The system of particles 3 and 4 are boosted backward to the momentum direction of particle-5.

Phase space is in the center-of-mass system and it is given by

$$\begin{aligned} d\tilde{\Gamma}_3 &= \frac{1}{2\pi} d\Gamma_2(1+2 \to 5+q) dQ^2 d\Gamma_2(q \to 3+4) \\ &= \frac{\beta(1+2;5,q)\beta(q;3,4)}{8^2(2\pi)^5} d\cos\theta_5 d\phi_5 dQ^2 d\cos\theta_{3,(34)} d\phi_{3,(34)}. \end{aligned}$$

To treat collinear singularity, new variables are introduced;

$$\frac{dQ^2}{Q^2} = d(\log Q^2),$$

then,

$$dQ^2 = Q^2 d(\log Q^2) = Q^2 \log(Q^2_{max}/Q^2_{min}) dt$$

where $Q^2 = Q^2_{min}(Q^2_{max}/Q^2_{min})^t$ for $0 < t < 1$. Moreover,

$$\begin{aligned} \cos\theta_3^* &= (\frac{\sqrt{Q^2}E_3}{E_3^*} - E_q)/P_q, \\ E_3^* &= \frac{Q^2 - m_5^2}{2\sqrt{Q^2}}, \\ E_q &= \frac{s + Q^2 - m_4^2}{2\sqrt{s}}, \\ P_q &= \sqrt{E_q^2 - Q^2}, \end{aligned}$$

where $\theta_3^*$ is polar angle of particle-3 in particle-3 and -4 rest frame, $E_3$ is an energy of particle-3 in a labframe. $E_3$ is used as integration variable instead of $\theta_3^*$, then

$$d(\cos\theta_3^*) = \frac{\sqrt{Q^2}}{E_3^* P_q} d(E_3).$$

Further modification

$$\begin{aligned} \frac{dE_3}{E_3} &= d(\log E_3), \\ &= \log(E_3^{max}/E_3^{min}) dt, \end{aligned}$$

for $0 < t < 1$ has been done.

**In the region $S_5$**

$4 \leftrightarrow 5$ of previous section.
**Options in `kinit.f`**

1. Physical parameters

   - `W` : Center of mass energy.
   - `COSCUT(1,1)` : Minimum of $\cos\theta_3$.
   - `COSCUT(2,1)` : Maximum of $\cos\theta_3$.
   - `COSCUT(1,2)` : Minimum of $\cos\theta_4$. Here, $\theta_4$ is the polar angle of particle-4 with respect to the beam axis.
   - `COSCUT(2,2)` : Maximum of $\cos\theta_4$.
   - `COSCUT(1,3)` : Minimum of $\cos\theta_5$. Here, $\theta_5$ is the polar angle of particle-5 with respect to the beam axis.
   - `COSCUT(2,3)` : Maximum of $\cos\theta_5$.
   - `ENGYCT(1,1)` : Minimum of $E_3$.
   - `ENGYCT(2,1)` : Maximum of $E_3$.
   - `ENGYCT(1,2)` : Minimum of $E_4$.
   - `ENGYCT(2,2)` : Maximum of $E_4$.
   - `ENGYCT(1,3)` : Minimum of $E_5$.
   - `ENGYCT(2,3)` : Maximum of $E_5$.
   - `AMASCT(1)` : Minimum of $Q$. $Q$ is the mass of the system of particles 4 and 5.
   - `AMASCT(2)` : Maximum of $Q$.
   - `ARESNS(1)` : Mass of resonance, this and the next parameter are meaningful only when `IRESN=+1`.
   - `ARESNS(2)` : Width of resonance.

2. Control parameters

   - `IRESN` : Treatment of $Q^2$.
     `= 0` : Treatment of $Q^2$. No resonance.
     `= 1` : $1/Q^2$ singularity.
     `= 2` : Narrow resonance.
     `= 3` : $1/Q^2$ singularity+Narrow resonance.

   - `ICOS4`: Collinear singularity of particle-4.
     `= 0`: No singularity.
     `= 1`: $\sim 1/t$ singularity, where $t = (p1 - p4)^2$.
     `= 2`: $\sim 1/t$ and $\sim 1/u$ singularity, where $u = (p1 - p5)^2$.

   - `ifrad`: Treatment of radiation.
     `= 1`: initial state radiation.
     `= 2`: final state radiation.
     `= 3`: initial+final state radiation.

### 2.8 2303

This is the kinematics for double-radiative processes. It is similar to 2302 and only the difference is that particle 3 and 4 are assumed to be photons. A energy ordering is required;

$$E_3 < E_4.$$

A statistical factor for idetical particles (1/2) is not needed.

### 2.9 2304

This is the kinematics for three-photon process. It is similar to 2302 and only the difference is that particle 3,4 and 5 are assumed to be photons. A energy ordering is required;

$$E_3 < E_4 < E_5.$$

A statistical factor for idetical particles (1/6) is not needed.

### 2.10 2401

This is the kinematics for 2 to 4 process in the center-of mass system. Here the final state first splits into the system of particles 3 and 4 and the system of particles 5 and 6. After that both system decays:

$$1 + 2 \longrightarrow q_1 + q_2 \quad , \quad q_1 \longrightarrow 3 + 4 \quad , \quad q_2 \longrightarrow 5 + 6$$

Integration variables are the polar angle $\theta$ and azimuthal angle $\phi$ for the first split, those angles for the two secondary splits, and the invariant masses of $q_1$ and $q_2$. Angles for the first split are defined with respect to the incoming particles and those for the secondary splits are defined in their own center-of-mass system with respect to the momentum direction of the system.

If there is a mass singularity for the two of particles in the final state, it is recommended to assign the two particles to form a pair above. Also user can introduce cutoff for angles and minimum energies as options.

The angles $\theta_{q1}$ and $\phi_{q1}$ represent the direction of the system of particles 3 and 4. Invariant masses are another variables.

$$Q_1^2 = q_1^2 = (p_3 + p_4)^2 \quad , \quad Q_2^2 = q_2^2 = (p_5 + p_6)^2$$

In the center of mass system of particles 3 and 4, angles $\theta_{3,(34)}$ and $\phi_{3,(34)}$ represent the direction of particle-3. Similarly, $\theta_{5,(56)}$ and $\phi_{5,(56)}$ are defined. The systems of particles 3 and 4,and 5 and 6 are boosted backward to the laboratory frame later.

Phase space is in the center-of-mass system and it is given by

$$\begin{aligned} d\tilde{\Gamma}_4 &= \frac{1}{(2\pi)^8} d\Gamma_2(1+2 \to q_1 + q_2) dQ_1^2 dQ_2^2 d\Gamma_2(q_1 \to 3+4) d\Gamma_2(q_2 \to 5+6) \\ &= \frac{\beta(1+2; q_1, q_2)\beta(q_1; 3, 4)\beta(q_2; 5, 6)}{8^3 (2\pi)^8} d\cos\theta_{q1} d\phi_{q1} \\ &\times \quad dQ_1^2 d\cos\theta_{3,(34)} d\phi_{3,(34)} dQ_2^2 d\cos\theta_{5,(56)} d\phi_{5,(56)}. \end{aligned}$$

**Options in `kinit.f`**

1. Physical parameters
   - `W` : Center of mass energy.
   - `COSCUT(1,1)` : Minimum of $\cos\theta_3$. This and angles below are all in the laboratory frame.

- `COSCUT(2,1)` : Maximum of $\cos\theta_3$.
- `COSCUT(1,2)` : Minimum of $\cos\theta_4$.
- `COSCUT(2,2)` : Maximum of $\cos\theta_4$.
- `COSCUT(1,3)` : Minimum of $\cos\theta_5$.
- `COSCUT(2,3)` : Maximum of $\cos\theta_5$.
- `COSCUT(1,4)` : Minimum of $\cos\theta_6$.
- `COSCUT(2,4)` : Maximum of $\cos\theta_6$.
- `ENGYCT(1,1)` : Minimum of $E_3$.
- `ENGYCT(2,1)` : Maximum of $E_3$.
- `ENGYCT(1,2)` : Minimum of $E_4$.
- `ENGYCT(2,2)` : Maximum of $E_4$.
- `ENGYCT(1,3)` : Minimum of $E_5$.
- `ENGYCT(2,3)` : Maximum of $E_5$.
- `ENGYCT(1,4)` : Minimum of $E_6$.
- `ENGYCT(2,4)` : Maximum of $E_6$.
- `AMASCT(1,1)` : Minimum of $Q_1$. $Q_1$ is the mass of the system of particles 3 and 4.
- `AMASCT(2,1)` : Maximum of $Q_1$.
- `AMASCT(1,2)` : Minimum of $Q_2$. $Q_2$ is the mass of the system of particles 5 and 6.
- `AMASCT(2,2)` : Maximum of $Q_2$.
- `ARESNS(1,1)` : Mass of resonance, this and the next parameter are meaningful only when `IRESNS(1)=+1`.
- `ARESNS(2,1)` : Width of resonance.
- `ARESNS(1,2)` : Mass of resonance, this and the next parameter are meaningful only when `IRESNS(2)=+1`.
- `ARESNS(2,2)` : Width of resonance.

2. Control parameters

- `IRESNS(1)` : Treatment of $Q_1^2$.
  `= 0` : No resonance.
  `= 1` : Narrow resonance.
  `= 2` : $1/Q_1^2$ singularity.

- `IRESNS(2)` : Treatment of $Q_2^2$.
  `= 0` : No resonance.
  `= 1` : Narrow resonance.
  `= 2` : $1/Q_2^2$ singularity.

- `ICOSQ3` : Treatment of $\theta_{q1}$.
  `= 0` : No singularity.
  `= 1` : 1/t singularity, where $t = (p1 - q1)^2$.
  `= 2` : 1/t+1/u singularity, where $u = (p1 - q2)^2$.

### 2.11 2402

This is the kinematics for 2 to 4 process in the center-of mass system. Here a particle 3 emits particle A and a particle 4 emits particle B. After that particles A and B collide into particles 5 and 6;

$$1 \longrightarrow 3 + A \quad , \quad 2 \longrightarrow 4 + B \quad , \quad A + B \longrightarrow 5 + 6$$

Integration variables are the polar angle $\theta$ and azimuthal angle $\phi$ of particles 3 and 4, those angles of particles 5 and 6 in their rest frame, energies of particles 3 and 4, and the invariant masses of 5 and 6. Angles of 3 and 4 are defined with respect to the incoming particles and those of 5 and 6 are defined in their own center-of-mass system with respect to the momentum direction of the system.

New variables are introduces as ;

$$\begin{aligned} Q_{1,2}^2 = -q_{1,2}^2 &= -(p_1 - p_{3,4})^2, \\ &= 2(EE_{3,4} \mp P_{1,2} P_{3,4} \cos\theta_{3,4}), \end{aligned}$$

and

$$q_{1,2}^0 = E - P_{3,4}.$$

Invariant mass is another variables.

$$Q_3^2 = q_3^2 = (p_5 + p_6)^2.$$

Phase space is in the center-of-mass system and it is given by

$$\begin{aligned} d\tilde{\Gamma}_4 &= \frac{1}{16(2\pi)^6} \frac{1}{P_3 P_4} \frac{1}{\sqrt{\Omega^2 - \Psi^2}} dQ_1^2 dQ_2^2 dq_1^0 dq_2^0 d\phi_3 \\ &\times \ \beta(q_1 q_2 : 56) dQ_1^2 d\cos\theta_{5,(56)} d\phi_{6,(56)}, \end{aligned}$$

where

$$\begin{aligned} \Psi &= W^2 - 2W(P_3 + P_4) + 2(E_3 E_4 - P_3 P4 \cos\theta_3 \cos\theta_4) \\ &\quad + m_3^2 + m_4^2 - Q_3^2, \\ \Omega &= 2E_3 E_4 \sin\theta_3 \sin\theta_4. \end{aligned}$$

**Options in `kinit.f`**

1. Physical parameters The same as 2401.
2. Control parameters
   - `IRESNS` : Treatment of $Q_3^2$. See description of 4001.
   - `ICOS3` : Treatment of $\theta_3$.
     `=0` : No singularity.
     `=1` : Mass singularity.
   - `ICOS4` : Treatment of $\theta_4$.
     `=0` : No singularity.
     `=1` : Mass singularity.

### 2.12 2403

This is a kinematics for the radiative processes. A particle 6 is assumed to be a photon. To treat both initial and final state radiations, a phase space with respect to photon angles is divided into four regions;

$$
\begin{aligned}
\sigma &= \int_S \frac{d\sigma}{d\Omega_4} d\Omega_4, \\
&= \int_{S_{12}} \frac{d\sigma}{d\Omega_4} d\Omega_4 + \int_{S_3} \frac{d\sigma}{d\Omega_4} d\Omega_4 + \int_{S_4} \frac{d\sigma}{d\Omega_4} d\Omega_4 + \int_{S_5} \frac{d\sigma}{d\Omega_4} d\Omega_4, \\
S_{12} &= \{\hat{P}_6 | min\{\theta_{16}, \theta_{26}, \theta_{36}, \theta_{46}, \theta_{56}\} = \theta_{16} \ or \ \theta_{26}\}, \\
S_3 &= \{\hat{P}_6 | min\{\theta_{16}, \theta_{26}, \theta_{36}, \theta_{46}, \theta_{56}\} = \theta_{36}\}, \\
S_4 &= \{\hat{P}_6 | min\{\theta_{16}, \theta_{26}, \theta_{36}, \theta_{46}, \theta_{56}\} = \theta_{46}\}, \\
S_5 &= \{\hat{P}_6 | min\{\theta_{16}, \theta_{26}, \theta_{36}, \theta_{46}, \theta_{56}\} = \theta_{56}\},
\end{aligned} \tag{5}
$$

where $\theta_{ij} = cos^{-1}(\hat{P}_i \cdot \hat{P}_i)$ and $\hat{P}_i$ is a unit (three) vector along a three momentum of a particle-$i$.

In each region, similar treatment as the `2402` is done to treat radiative photon.

**Options in `kinit.f`**

1. Physical parameters

   - `W` : Center of mass energy.
   - `COSCUT(1,1)` : Minimum of $\cos\theta_3$. This and angles below are all in the laboratory frame.
   - `COSCUT(2,1)` : Maximum of $\cos\theta_3$.
   - `COSCUT(1,2)` : Minimum of $\cos\theta_4$.
   - `COSCUT(2,2)` : Maximum of $\cos\theta_4$.
   - `COSCUT(1,3)` : Minimum of $\cos\theta_5$.
   - `COSCUT(2,3)` : Maximum of $\cos\theta_5$.
   - `COSCUT(1,4)` : Minimum of $\cos\theta_6$.
   - `COSCUT(2,4)` : Maximum of $\cos\theta_6$.
   - `ENGYCT(1,1)` : Minimum of $E_3$.
   - `ENGYCT(2,1)` : Maximum of $E_3$.
   - `ENGYCT(1,2)` : Minimum of $E_4$.
   - `ENGYCT(2,2)` : Maximum of $E_4$.
   - `ENGYCT(1,3)` : Minimum of $E_5$.
   - `ENGYCT(2,3)` : Maximum of $E_5$.
   - `ENGYCT(1,4)` : Minimum of $E_6$.
   - `ENGYCT(2,4)` : Maximum of $E_6$.
   - `AMASCT(1)` : Minimum of $Q$. $Q$ is the mass of the system of particles 4 and 5.
   - `AMASCT(2)` : Maximum of $Q$.
   - `ARESNS(1,1)` : Mass of first resonance.
   - `ARESNS(2,1)` : Width of first resonance.

- `ARESNS(1,2)` : Mass of second resonance.
- `ARESNS(2,2)` : Width of second resonance.

2. Control parameters

    - `IRESNS` : Treatment of $Q^2$.
      `= 0` : no-singularity
      `= 1` : narrow resonance (single)
      `=-1` : narrow resonance (single)+1/Q2 singularity
      `= 2` : narrow resonance (double)
      `=-2` : narrow resonance (double)+1/Q2 singularity
      `= 3` : Q2=S peak
      `=-3` : 1/Q2 singularity only
      `= 4` : narrow resonance (single)+Q2=S peak

    - `ICOS3` : Treatment of $\theta_3$.
      `= 0` : no-singularity
      `= 1` : 1/t singularity
      `= 2` : 1/t + 1/u singularity

    - `ICOS4` : Treatment of $\theta_4$.
      `= 0` : no-singularity
      `= 1` : z-axis=-p3 ; flat rapidity
      `=-1` : z-axis=p1 ; 1/t singularity
      `=-2` : z-axis=p1 ; 1/t + 1/u singularity

    - `ifrad`: Treatment of radiation.
      `= 1`: initial state radiation.
      `= 2`: final state radiation.
      `= 3`: initial+final state radiation.