One- and Two-Loop Four-Point Integrals with XLOOPS-GiNaC

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Outline

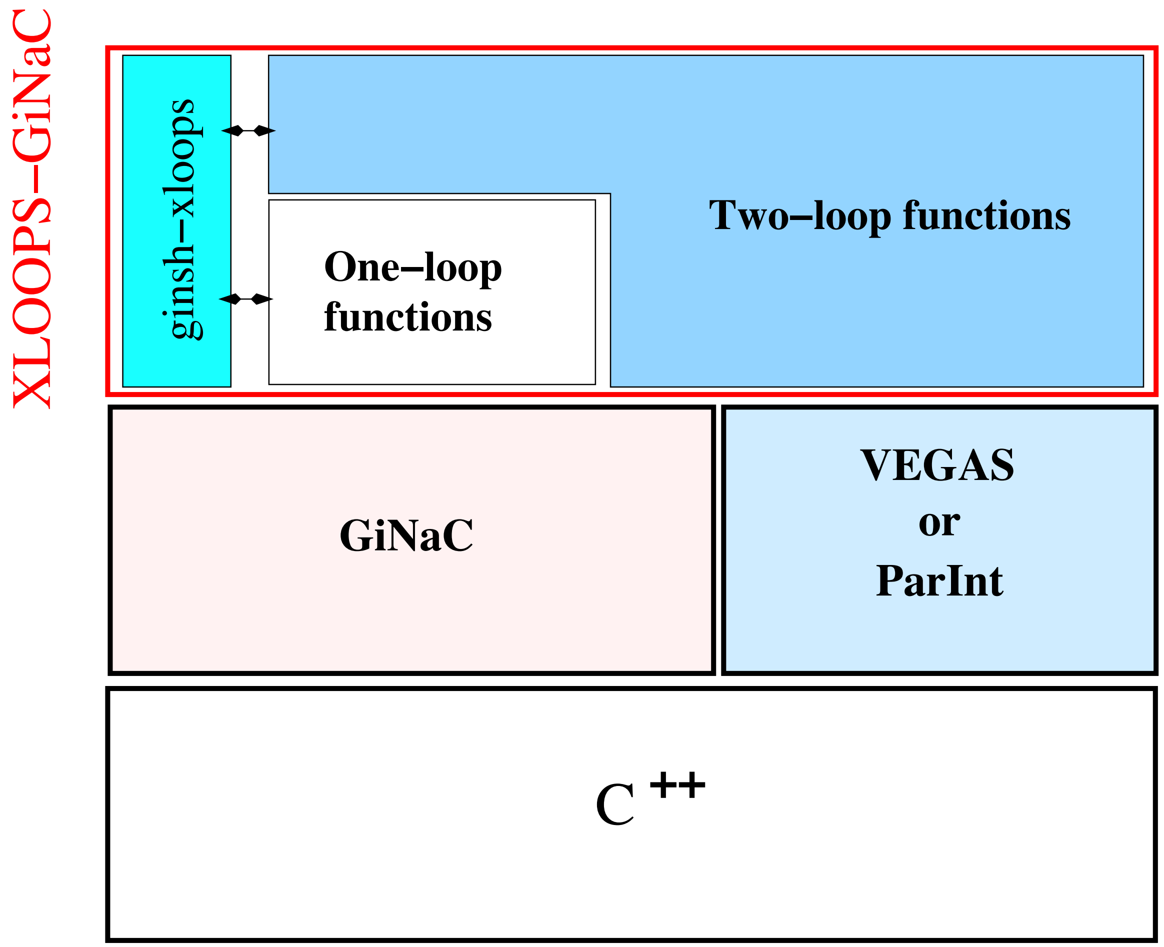
- A brief history of XLOOPS-GiNaC
- Box integrals in Parallel-Orthogonal Space (POS)
- General procedure for box integration
- One-loop box results
- Two-loop box results (preliminary)

A Brief History of XLOOPS-GiNac

- 1991-1994: Loop Integration in the Parallel - Orthognal (POS) for one-loop and two-loop was introduced by D. Kreimer.
- 1995-1998: Xloops was developed based on MapleV (Brücher, Franzkowski, Frink, Kreckel, Kreimer)
- Since 1999: GiNaC was developed to replace MapleV (Bauer, Frink, Kreckel, Vollinga,....)
- Since 1999: XLOOPS-GiNaC was developed (Bauer, Do, and Knodel, Seul, Spiesberger, Vollinga...)
- About 20 related articles was published

Structure and Features

- Simple tensor reduction procedure
- Valid for massless and/or arbitrary massive particles and arbitrary tensor order



- Two-loop Two-point integrals can be integrated by two-fold numerical integration using pVegas or ParInt.
- All code is in C^{++}
- http://wwwthep.physik.uni-mainz.de/~xloops/

Pros and Cons

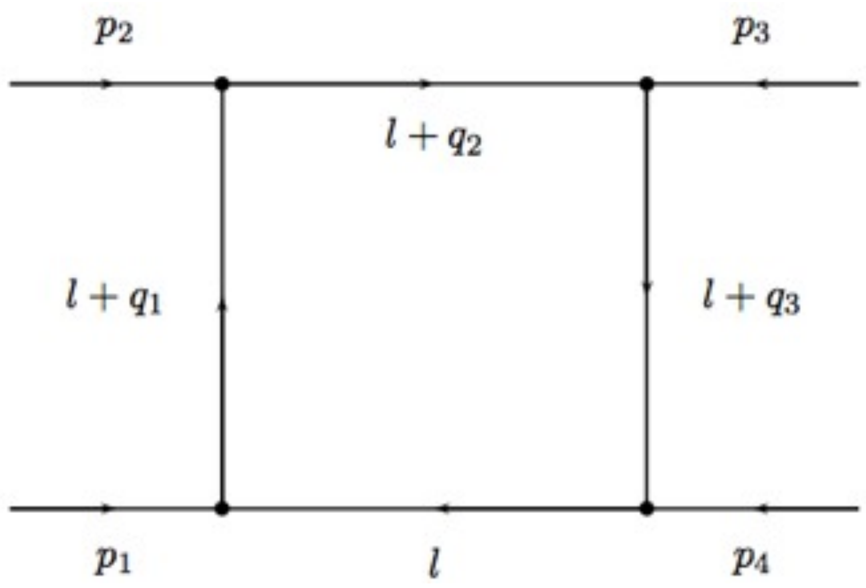
- Gram determinant problem free.

  Tensor integrals are automatically decomposed into integrals of components of internal momenta. No matrix inversion!

- The price is Lorentz invariant structure of the integrals.

- The package has never reached a productive status. Tests should be done!

Box integrals in POS



$$q_1 = q_1(q_{10},0,0,0) = p_1$$

$$q_2 = q_2(q_{20},q_{21},0,0) = p_1 + p_2$$

$$q_3 = q_3(q_{30},q_{11},q_{32},0) = p_1 + p_2 + p_3$$

.... = ...

$$q_{10} = \sqrt{p_1^2} \quad , \quad q_{20} = \frac{p_1 p_2}{\sqrt{p_1^2}} + \sqrt{p_1^2}$$

$$q_{21} = \sqrt{\left(\frac{p_1 p_2}{\sqrt{p_1^2}} + \sqrt{p_1^2}\right)^2 - (p_1+p_2)^2} \quad , \quad q_{30} = \frac{p_1 p_3}{\sqrt{p_1^2}} + \frac{p_1 p_2}{\sqrt{p_1^2}} + \sqrt{p_1^2}$$

$$q_{31} = \frac{(p_1 p_2)(p_1 p_3) - (p_1 p_2)^2 - p_1^2 p_2^2 - p_1^2 (p_2 p_3)}{q_{21}\sqrt{p_1^2}} \quad , \quad q_{32} = \sqrt{q_{30}^2 - q_{31}^2 - p_4^2}$$

One-loop Box Integrals

- Scalar one-loop box integral in 4-dim POS

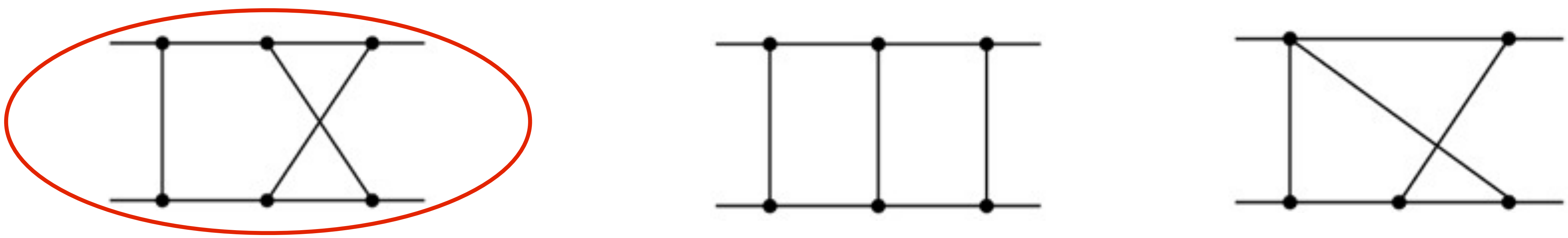
$$D_0 = 2 \int_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int_{0}^{\infty} dl_\perp \frac{1}{\prod_{k=1}^{4} \left[(l + q_k)^2 - m_k^2 + i\rho \right]}$$

$$(l + q_i)^2 \to (l_0 + q_{i0})^2 - (l_1 + q_{i1})^2 - (l_2 + q_{i2})^2 - l_\perp^2$$

- This is an integral over 4 variables, symmetric in $l_\perp$
- The masses m_k can be complex or real

Two-loop Box Integrals

- We are interested in a subgroup of Scalar Two-loop box integral (crossed and planar topologies)



(a) (b) (c)

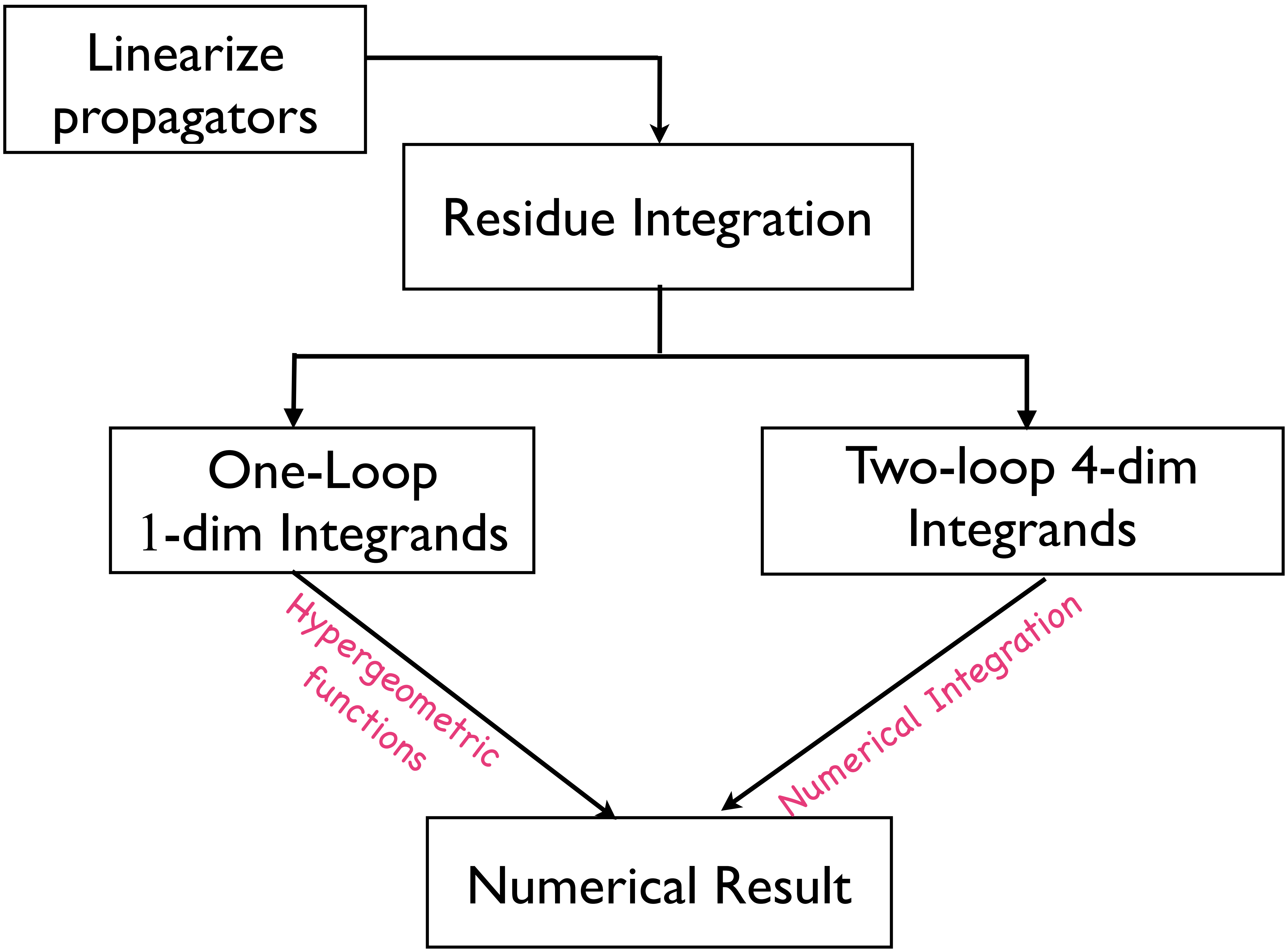
$$S_0 = 4 \int_0^{\infty} dl_\perp dk_\perp \int_{-\infty}^{\infty} \frac{dl_0 dl_1 dl_2 dk_0 dk_1 dk_2}{P(l)P(l+k)P(k)}$$

$$P(l) \quad = \quad \prod_i^{N_l} \left[(l+q_i)^2 - m_i^2 + i\eta\right] \ , \ P(k) = \prod_n^{N_k} \left[(k+q_n)^2 - m_n^2 + i\eta\right]$$

$$P(l+k) = \prod_j^{N_{lk}} \left[(l+k+q_j)^2 - m_j^2 + i\eta\right]$$

Kreimer Phys.Lett.B347, 1995; Kreckel, Eur.Phys.J.C6 1998

Algorithm


Linearize propagators
Residue Integration
One-Loop 1-dim Integrands
Two-loop 4-dim Integrands
Hypergeometric functions
Numerical Integration
Numerical Result

Linearize propagators

- For arbitrary masses, the poles of the integrand in the complex planes of l_3, k_3, (l_3+k_3) stay on the first and the third quadrant.

$$
\begin{aligned}
P_i(l) &= (l_0 + q_{i0})^2 - \cdots - l_3^2 - m_i^2 + i\eta \\
P_n(l+k) &= (l_0 + k_0 + q_{n0})^2 - \cdots - (l_3 + k_3)^2 - m_n^2 + i\eta \\
P_j(k) &= (k_0 + q_{j0})^2 - \cdots - k_3^2 - m_j^2 + i\eta
\end{aligned}
$$

- 3 of 4 variables (for one-loop) and 4 of 8 variables (for 2-loop) can be linearized by a set of transformations
  - *Metric rotation:* (+,-,-,-) $\longrightarrow$ (+, -, -, +)
  - $l_0 \longrightarrow l_0 + l_1$, $k_0 \longrightarrow k_0 + k_1$, $l_3 \longrightarrow l_3 + l_2$, $k_3 \longrightarrow k_3 + k_2$

Kreimer et al: Phys.Lett.B347, 1995,
Kreckel Eur.Phys.J.C6 1998, Franzkowski thesis 1998

- For arbitrary complex masses, the pole's locations are more complicated but the above transformations still applicable!

One-loop 1-dim box integrals

$$
\begin{aligned}
D_0 \;=\; & \bigoplus_{nmlk} |1-\beta_{mlk}\varphi_{mlk}| \times \\
& \Bigg[\int_0^{\infty} dz\, G(z) \Bigg\{ (f_{lk}g_{mlk} + f^{-}_{lk}g_{mlk}) \ln\left(\frac{F}{\beta}\right) \\
& - f_{lk}g_{mlk} \ln\left(\frac{(1-\beta\varphi)z+F}{\beta}\right) - f_{lk}g^{-}_{mlk} \ln\left(-\frac{(1-\beta\varphi)z+F}{\beta}\right) \\
& - (f_{lk}g_{mlk} + f^{-}_{lk}g_{mlk}) \ln\left(\frac{-\frac{P}{\beta}z^2 + (E-\frac{Q}{\beta})z - m_k^2 + i\varrho}{Q+Pz}\right) \\
& + f_{lk}g_{mlk} \ln\left(\frac{-P\varphi z^2 + (E-Q\varphi)z - m_k^2 + i\varrho}{Q+Pz}\right) \\
& + f_{lk}g^{-}_{mlk} \ln\left(-\frac{-P\varphi z^2 + (E-Q\varphi)z - m_k^2 + i\varrho}{Q+Pz}\right) \Bigg\} \\
& + \int_{-\infty}^{0} dz\, G(z) \Bigg\{ -f^{-}_{lk}g^{-}_{mlk} \ln\left(\frac{F}{\beta}\right) - f_{lk}g^{-}_{mlk} \ln\left(-\frac{F}{\beta}\right) \\
& + (f^{-}_{lk}g^{-}_{mlk} + f^{-}_{lk}g_{mlk}) \ln\left(\frac{(1-\beta\varphi)z+F}{\beta}\right) \\
& + f^{-}_{lk}g^{-}_{mlk} \ln\left(\frac{-\frac{P}{\beta}z^2 + (E-\frac{Q}{\beta})z - m_k^2 + i\varrho}{Q+Pz}\right) \\
& + f_{lk}g^{-}_{mlk} \ln\left(-\frac{-\frac{P}{\beta}z^2 + (E-\frac{Q}{\beta})z - m_k^2 + i\varrho}{Q+Pz}\right) \\
& - (f^{-}_{lk}g^{-}_{mlk} + f^{-}_{lk}g_{mlk}) \ln\left(\frac{-P\varphi z^2 + (E-Q\varphi)z - m_k^2 + i\varrho}{Q+Pz}\right) \Bigg\} \Bigg]
\end{aligned}
$$

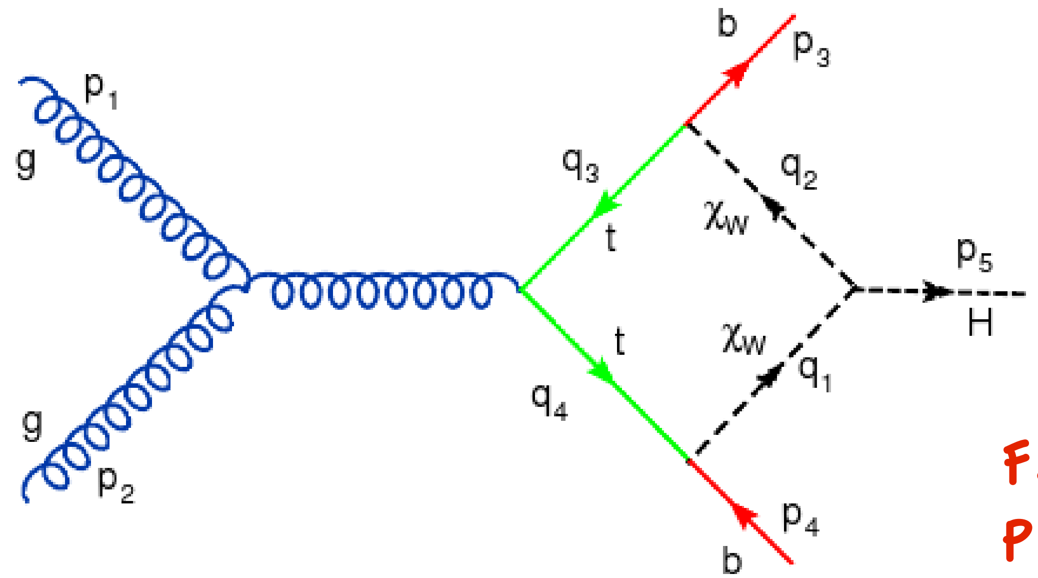
Two-loop 2-dim box integrals

- Including of 32 terms of the form

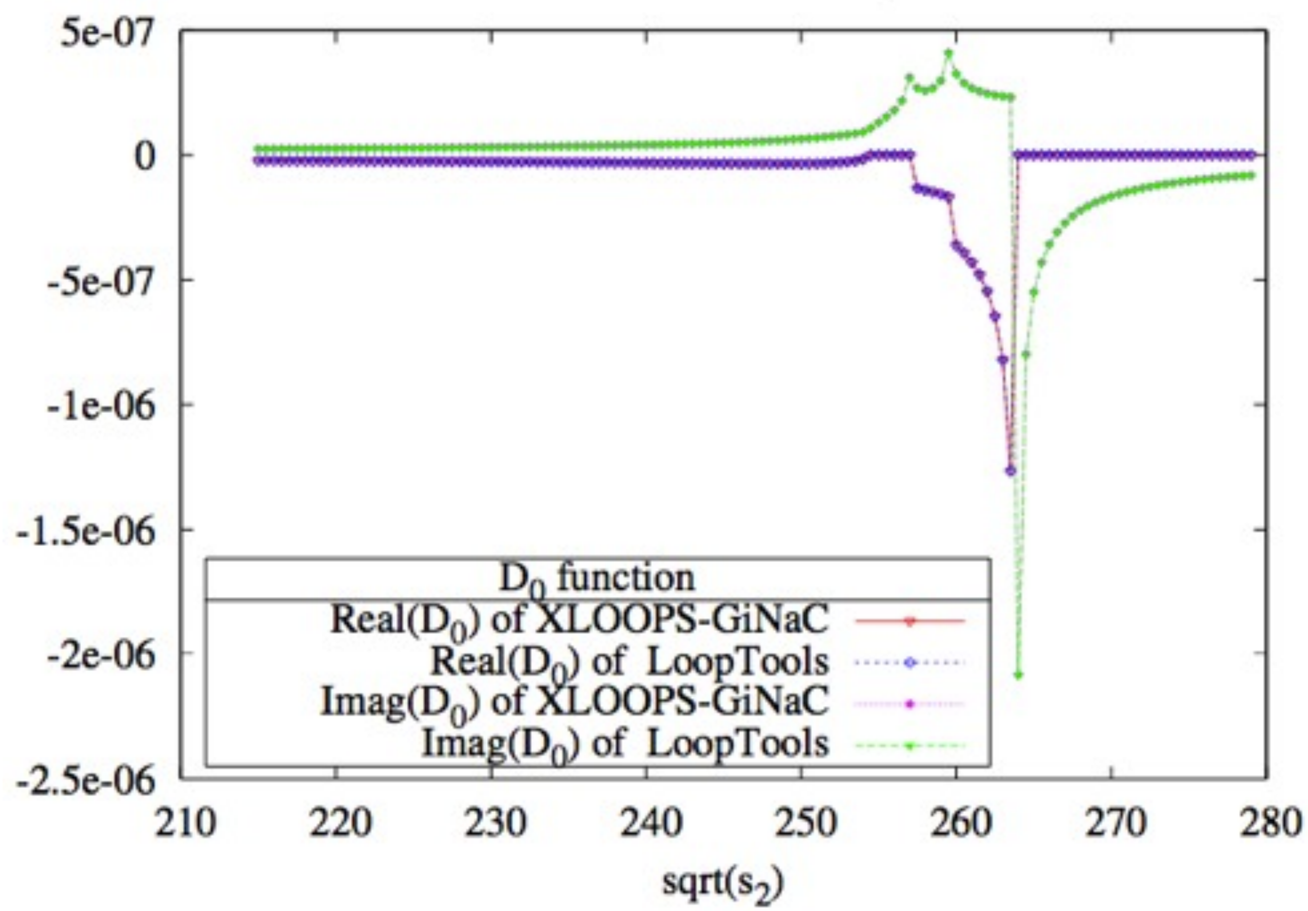
$$\sum_{g=1}^{N_g}\sum_{g'\neq g}^{N_g}\sum_{l=1}^{N_l}\sum_{l'\neq l}^{N_l}\int_V dl_0 dl_3 dk_0 dk_3 \frac{Q(l_0,l_3;k_0,k_3)}{\prod_{l''\neq l,l'}^{N_l} D_{l'}(l_0,l_3;k_0,k_3)\prod_{k=1}^{N_k} D_{k'}(l_0,l_3;k_0,k_3)}$$

- *Q* is polynomial of l_0, l_3, k_0, k_3
- *D*'s are quadratic in l_0, l_3, k_0, k_3
- *V* is a finite volume which is constrained by the locations of poles.
- Integrand is a huge expression ➟ Numerical stability and performance problems!!!!

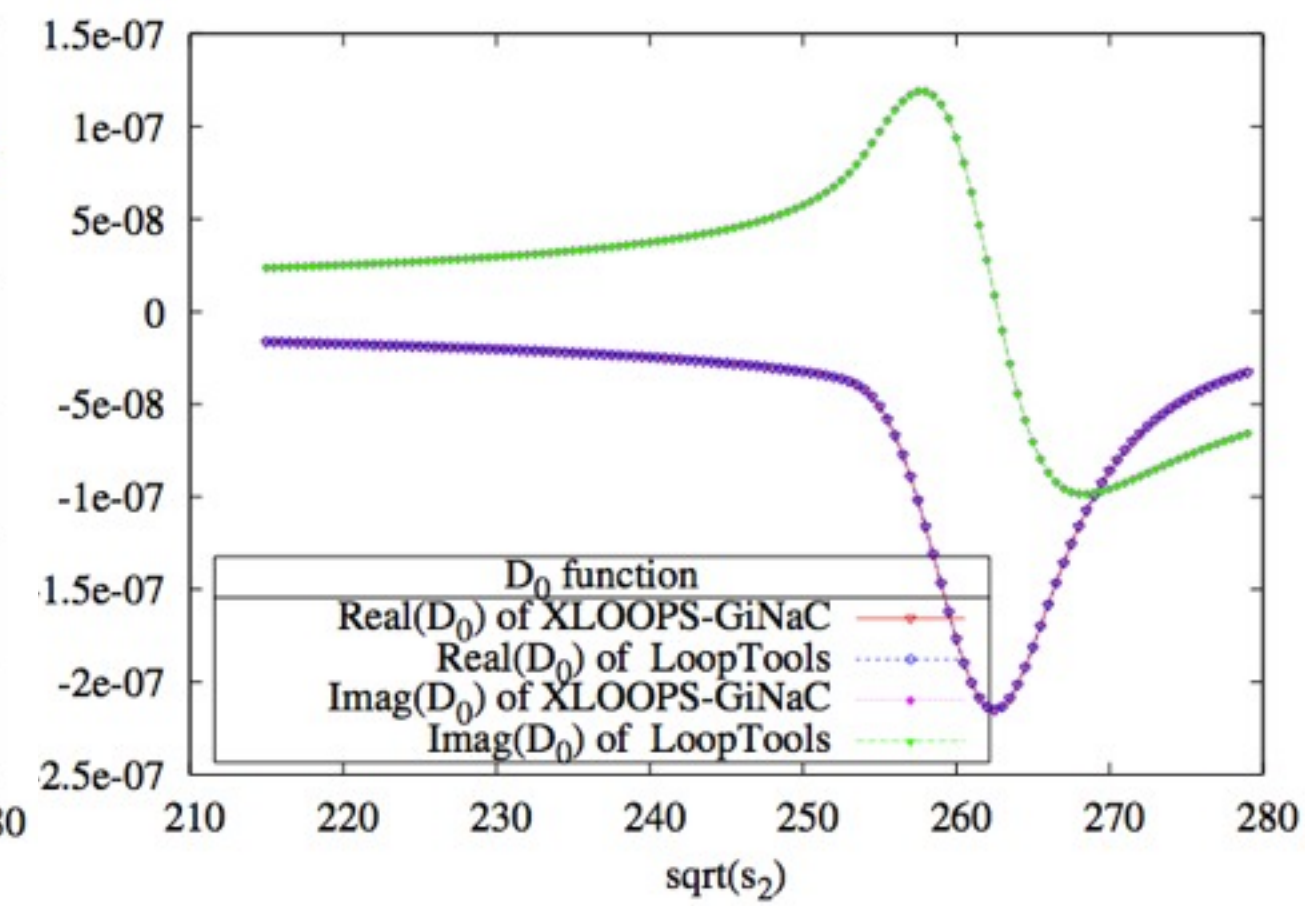
Numerical Results - D_0



F. Boudjema, Le Duc Ninh, Phys.Rev.D78:093005,2008



Zero widths

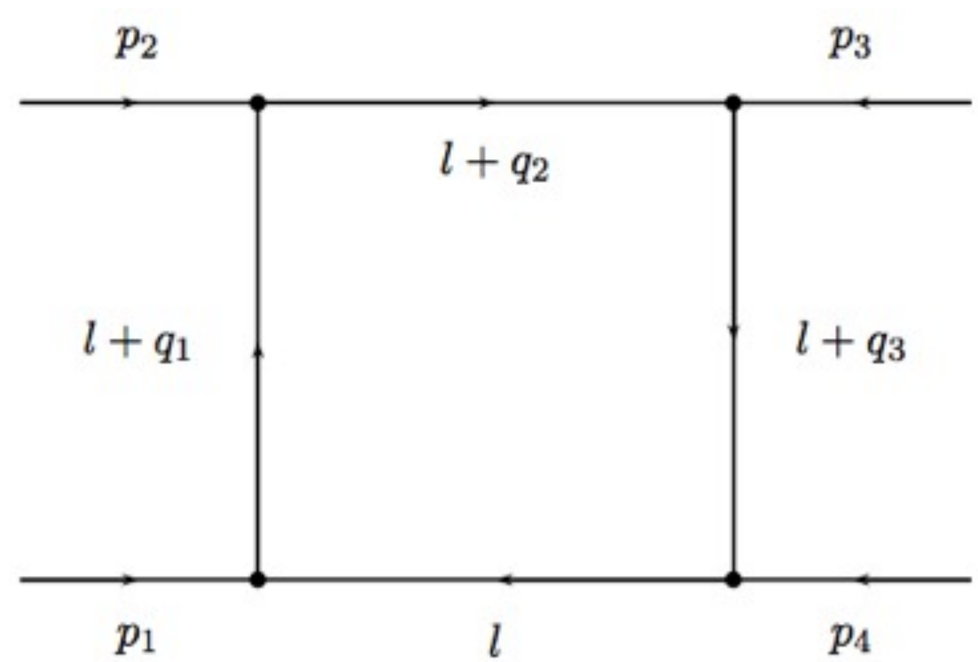


Γ_t =1.5 (GeV), Γ_W =2.1 (GeV)

Numerical Results - D0

Le Duc Ninh, Dao Thi Nhung,
Comput.Phys.Commun.180, 2009

p_4^2		XLOOPS-GiNaC	LoopTools 2.5
40	Real:	$3.179575942209314007E-6$	$3.17958e^{-06}$
	Imag:	$1.069504655916838083 5E-7$	$1.0695e^{-07}$
20	Real:	$3.160949561047315089 2E-6$	$3.16095e^{-06}$
	Imag:	$1.061520120481490588 8E-7$	$1.06152e^{-07}$
0	Real:	$3.142631279651899166E-6$	$3.14263e^{-06}$
	Imag:	$1.053697251811589911 7E-7$	$1.0537e^{-07}$
-20	Real:	$3.124612297957250085E-6$	$3.12461e^{-06}$
	Imag:	$1.046030580115410743 7E-7$	$1.04603e^{-07}$
-40	Real:	$3.106884176506483581 6E-6$	$3.10688e^{-06}$
	Imag:	$1.038514894584754838E-7$	$1.03851e^{-07}$



$$D_0(10, -60, -10, p_4^2, 200, -10, 100-5i, 200-2i, 300-3i, 400-4i)$$

Numerical Results - S_{0a}

Numerical stability might turn out to be finally an issue (Richard Kreckel)

Outlooks

- Numerical results for Two-loop box
- Clean the code and make a better configuration script for XLOOPS-GiNaC
- More tests

Thank you!

Backup

$$
\begin{aligned}
D_0 \;=\; \bigoplus_{nmlk} \frac{|1-\beta_{mlk}\varphi_{mlk}|}{P_{mlk}} \times \qquad\qquad (104)\\
\Bigg\{ & -(f_{lk}+f_{\bar{lk}})g_{mlk}\left(\ln\left(\frac{F}{\beta}\right) - \mathit{Eta}\left(P\sigma Z_1, Z_2;\beta\right)\right) GZ(-T_1,-T_2;0)\\
& -f_{lk}(g_{mlk}+g_{\bar{mlk}})\;\mathit{Eta}\left(P\sigma Z_1, Z_2;\phi\right)\; GZ(-T_1,-T_2;0)\\
& -g_{\bar{mlk}}\left[f_{\bar{lk}}\ln\left(\frac{F}{\beta}\right) + f_{lk}\ln\left(-\frac{F}{\beta}\right)\right] GZ(T_1,T_2;0)\\
& +g_{\bar{mlk}}(f_{lk}+f_{\bar{lk}})\;\mathit{Eta}\left(-P\sigma Z_1,-Z_2;\beta\right) GZ(T_1,T_2;0)\\
& -f_{\bar{lk}}(g_{mlk}+g_{\bar{mlk}})\;\mathit{Eta}\left(-P\sigma Z_1,-Z_2;\phi\right) GZ(T_1,T_2;0)\\
& -f_{lk}g_{mlk}\;\mathcal{L}^{+}\left(\frac{1-\beta\phi}{\beta},\frac{F}{\beta}\right) - f_{lk}g_{\bar{mlk}}\;\mathcal{L}^{+}\left(-\frac{1-\beta\phi}{\beta},-\frac{F}{\beta}\right)\\
& +f_{\bar{lk}}(g_{mlk}+g_{\bar{mlk}})\;\mathcal{L}^{-}\left(-\frac{1-\beta\phi}{\beta},\frac{F}{\beta}\right)\\
& -(f_{lk}+f_{\bar{lk}})g_{mlk}\;\left\{\mathcal{L}^{+}\left(P\sigma,-P\sigma Z_1|_{\beta}\right)+\mathcal{L}^{+}\left(1,-Z_2|_{\beta}\right)\right\}\\
& +f_{lk}(g_{mlk}+g_{\bar{mlk}})\;\left\{\mathcal{L}^{+}\left(P\sigma,-P\sigma Z_1|_{\phi}\right)+\mathcal{L}^{+}\left(1,-Z_2|_{\phi}\right)\right\}\\
& +f_{\bar{lk}}g_{mlk}\;\left\{\mathcal{L}^{+}\left(P,Q\right)+\mathcal{L}^{-}\left(-P,Q\right)\right\} - f_{lk}g_{\bar{mlk}}\;\left\{\mathcal{L}^{+}\left(-P,-Q\right)+\mathcal{L}^{-}\left(P,-Q\right)\right\}\\
& +(f_{lk}+f_{\bar{lk}})g_{\bar{mlk}}\;\left\{\mathcal{L}^{-}\left(P\sigma,P\sigma Z_1|_{\beta}\right)+\mathcal{L}^{-}\left(1,Z_2|_{\beta}\right)\right\}\\
& -f_{\bar{lk}}(g_{mlk}+g_{\bar{mlk}})\;\left\{\mathcal{L}^{-}\left(P\sigma,P\sigma Z_1|_{\phi}\right)+\mathcal{L}^{-}\left(1,Z_2|_{\phi}\right)\right\}\\
& +\int_0^{\infty} dz\,\bar{G}(z)\left\{f_{lk}g_{\bar{mlk}}\,\eta\left(S(z),-\frac{1}{Pz+Q}\right) - f_{\bar{lk}}g_{mlk}\,\eta\left(S(z),\frac{1}{Pz+Q}\right)\right\}\\
& +\int_0^{\infty} dz\,\bar{G}(-z)\left\{f_{lk}g_{\bar{mlk}}\,\eta\left(S(-z),\frac{1}{Pz-Q}\right) - f_{\bar{lk}}g_{mlk}\,\eta\left(S(-z),\frac{1}{Q-Pz}\right)\right\}\;\Bigg\}
\end{aligned}
$$

Backup

$$\mathcal{L}^{+}(a,b,T_1,T_2) = \frac{1}{T_1 - T_2}\left\{Li_2(1+\frac{a}{b}T_1) - Li_2(1+\frac{a}{b}T_2) + \eta(-T_1, a/b)\ln(1+\frac{a}{b}T_1) - \eta(-T_2, a/b)\ln(1+\frac{a}{b}T_2)\right\}$$

$$\int_{-\infty}^{a} \bar{G}(z)dz = \int_{-\infty}^{a} dz \frac{-1}{(z-T_1)(z-T_2)} = -\frac{\ln(T_1 - a) - \ln(T_2 - a)}{(T_1 - T_2)}$$

$$\bigoplus_{nmlk} = i\pi^2 \sum_{k=1}^{4} \sum_{\substack{l=1 \\ l\neq k}}^{4} \sum_{\substack{m=1 \\ m\neq l \\ m\neq k}}^{4} \frac{[1-\delta_{lk}(AC_{lk})][1-\delta(B_{mlk})]}{AC_{lk}[B_{mlk}A_{nlk} - B_{nlk}A_{mlk}]}$$

Backup

Techniques using Mellin-Barnes transformation are very successfully applied to at least integrals with few masses[.....]and it seems to be the best method around. (Jens Vollinga)

Tensor decomposition

For J-parallel dimension cases $q_\mu = (q_0, \ldots q_J, \vec{0})$

$$k^\mu q_\mu = (k_0 + q_{i0})^2 + \cdots + (k_J + q_{iJ})^2$$

$$k^\mu = k_\perp^\mu + \sum_i^J f(k \cdot q_i) q_i^\mu$$

$$N_i(k) = (k_0 + q_{i0})^2 + \cdots + (k_J + q_{iJ})^2 + \vec{k}_\perp^2 - m_i^2 + i\eta$$

$$k^{\mu_1} \ldots k^{\mu_N} \to \sum k_\perp^{\mu_1} \ldots k_\perp^{\mu_i} Q(k_0, \ldots, k_J) q^{\mu_{i+1}} q^{\mu_N}$$

The integrals are symmetric in $k_\perp^\mu$ ➠ one ends up with integrals in $k_0, \ldots, k_J$