Numerical calculation of one-loop integration with Hypergeometric functions

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1 Introduction

■ Purpose:

- To handle arbitrary combination of mass parameters in d-dimensional 1-loop calculations.

  Unified calculation of massive and massless cases.

  Tensor integral

  $\Leftarrow$ differentiation in terms of mass parameters.

- To clarify the singular structure of 1 loop integrations.

- To avoid numerical cancellation in 1 loop calculations.

$\Rightarrow$ Hypergeometric functions will be useful.

Introduction

■ Representation with hypergeometric functions:

- Regge (1969, a class of generalized hypergeometric equations)
- Tarasov et al., Davydychev, Kalmykov, ... (1-, 2-loop, ...)
- Duplančić and Nižić, Kurihara (1-loop, for massless QCD with IR)

■ This work:

New analytic calculation using Lauricella's F_D (a multi-variable extension of Gauss' F) with sample numerical calculations.

Introduction

■ Classes of Hypergeometric functions:

- Gauss' hypergeometric functions F (1 variable).
- Generalized hypergeometric functions ${}_pF_q$ (1 variable).
- Appell's F_1, F_2, ... (2 variables).
- Lauricella's F_D (n variables). (includes Gauss' F and Appell's F_1 for special cases).
- Aomoto, Gelfand, GKZ (Gelfand, Zelevinsky and Karpanov)

Introduction

- Properties of F and many of other hypergeometric functions:
  - Analytic structures is known ($z = 0, 1, \infty$ for F).
  - Power series representations is known.
  - Integral forms are know.
  - Differential equation is known.
  - A rich set of identities is known.
  - Many asymptotic behaviors are known.
  - However, numerical calculations are not trivial for any values of parameters and variables.

    $\Rightarrow$ We only need for a limited combinations of parameters.

Introduction

■ Examples of expansion for $d = 4 - \epsilon \to 4 + 0$:

$$F(1, \epsilon; 1 + \epsilon; z) = 1 - \sum_{k=1}^{\infty} (-1)^k \operatorname{Li}_k(z) \epsilon^k$$

$$= 1 - \epsilon \log(1 - z) - \epsilon^2 \operatorname{Li}_2(z) + O(\epsilon^3),$$

$$F(a\epsilon, b\epsilon; 1 + b\epsilon; z) = 1 + ab\epsilon^2 \operatorname{Li}_2(z) + O(\epsilon^3).$$

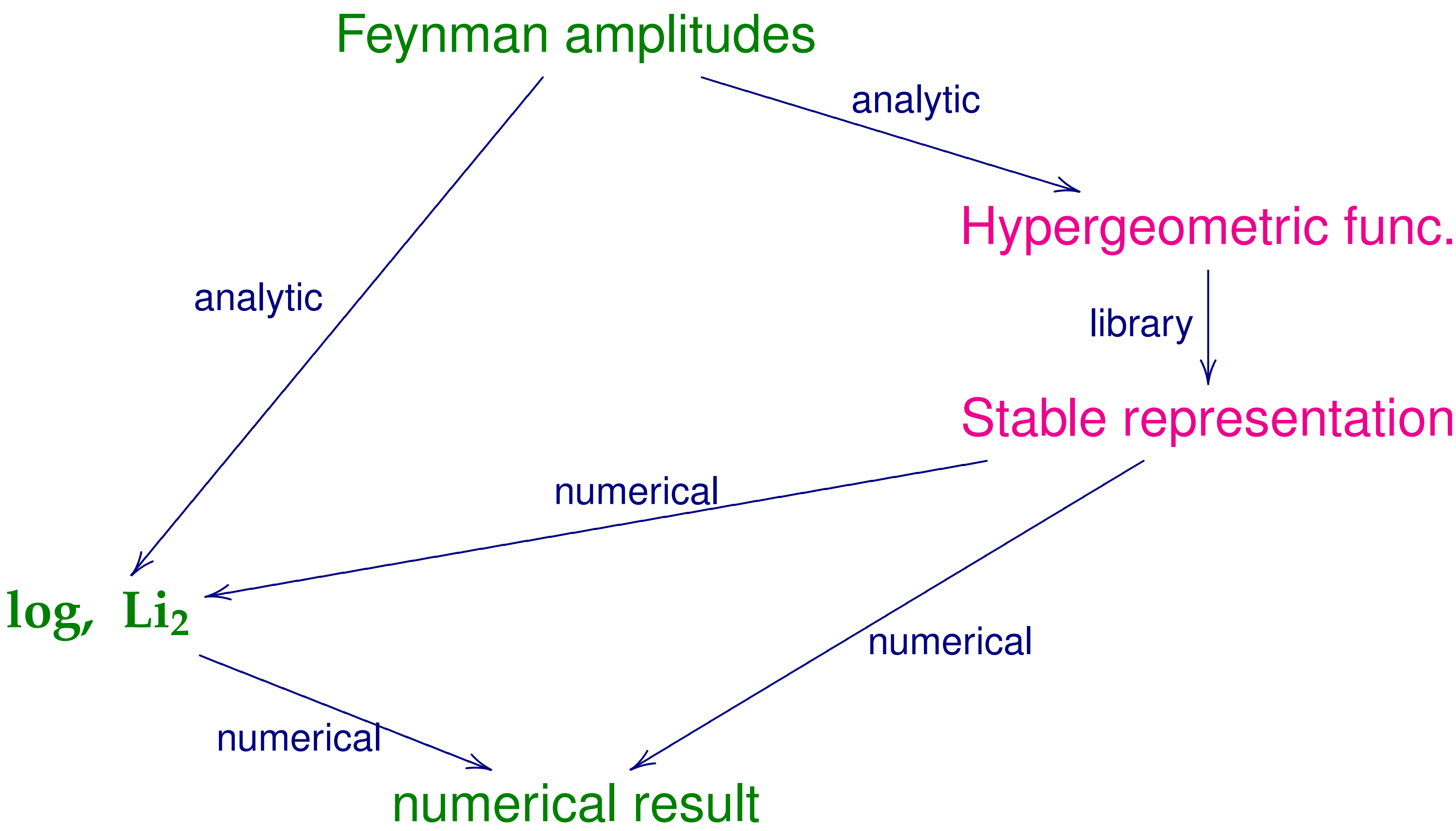
■ In many cases, F_D is expressed by a combination of F, which reduces to $\log$ and Li_2. Expression is not unique with may identities (with different numerical stabilities).

⇒ numerically calculable.

⇒ a library will be able to select a stable expression automatically using identities and asymptotic forms of F_D and/or F.

Introduction

- Plan



Introduction

■ Analytic part

Calculate

1. For the general case with arbitrary mass parameters, kinematic parameters and the dimension.
2. For tensor integration : differentiation in terms of mass parameters.
3. For special cases : by taking a massless limit or other limits.

■ Our analytic results :

- 2-point function : naturally expressed by F_1.
- 3-point function : expressed by F_D.
- 4-point function : expressed by F_D up to order $1 = \epsilon^0$ ($d = 4 - \epsilon$) including IR poles.

Introduction

■ The order to take limits (they are not commutable).

Results depends of the order to take limits among $m_j \to 0$, $\epsilon \to 0$ and $s, t \to$ (specific values) around singularities.

1. Differentiation in terms of m_j for tensor integration.
2. If particles are massless, take the limit $m_j \to 0$ before $d \to 4$.
   (massless particles in d-dimensional space-time)
3. $\epsilon \to 0$, or expansion of the results in terms of ϵ, before the numerical calculation.
4. Kinematic variables s, t etc. are substituted by specific values for the numerical calculation.

2 2-point function

2.1 General case

- 2-point scalar integration for general parameters.

$$
\begin{aligned}
I_2^{(\alpha)} &= \int_0^\infty dx_1 \int_0^\infty dx_2 \ \delta(1 - x_1 - x_2) \ \mathcal{D}^\alpha, \\
\mathcal{D} &= -p^2 x_1 x_2 + m_1^2 x_1 + m_2^2 x_2 - i\varepsilon
\end{aligned}
$$

- Scalar integration: $\alpha = -\epsilon > 0$ around 4 dimensional space-time.
- Tensor integration: ($\alpha = -\epsilon + k_1 + k_2$)

$$
\frac{\partial^{k_1}}{\partial(m_1^2)^{k_1}} \frac{\partial^{k_2}}{\partial(m_2^2)^{k_2}} I_2^{(\alpha)} \propto \int_0^\infty dx_1 \int_0^\infty dx_2 \ \delta(1 - x_1 - x_2) \ x_1^{k_1} x_2^{k_2} \mathcal{D}^{\alpha - k_1 - k_2}.
$$

2-point function

- Integrate over x_2 with δ-function ($x_1 = x,\ x_2 = 1 - x$).

$$I_2^{(\alpha)} = \int_0^1 \mathcal{D}^\alpha\ dx,$$

$$\mathcal{D} = p^2x^2 + [-p^2 + m_1^2 - m_2^2]x + m_2^2 = \mathcal{D}(0)\left(1 - \frac{x}{y^+}\right)\left(1 - \frac{x}{y^-}\right),$$

$$y^\pm = \frac{p^2 - m_1^2 + m_2^2 \pm \sqrt{D}}{2p^2} \in \mathbb{C}, \quad D = (-p^2 + m_1^2 + m_2^2)^2 - 4m_1^2m_2^2.$$

- Thus

$$I_2^{(\alpha)} = (\mathcal{D}(0))^\alpha \int_0^1 \left(1 - \frac{x}{y^+}\right)^\alpha \left(1 - \frac{x}{y^-}\right)^\alpha\ dx.$$

$\Rightarrow$ Appell's F_1.

2-point function

- An integral representation of Appell's F_1:

$$F_1(\alpha,\beta,\beta';\gamma;y,z)$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 x^{\alpha-1}(1-x)^{\gamma-\alpha-1}(1-yx)^{-\beta}(1-zx)^{-\beta'}\,dx.$$

This function is regular except for $y = 0, 1, \infty,\ z = 0, 1, \infty,\ y = z$.

F_1 is a generalization of Gauss' F:

$$F_1(\alpha,\beta,0;\gamma;y,z) = F(\alpha,\beta;\gamma;y).$$

2-point function

- Comparing two integrations, we obtain

$$I_2^{(\alpha)} = \mathcal{D}(0)^\alpha F_1(1, -\alpha, -\alpha; 2; \frac{1}{\gamma^+}, \frac{1}{\gamma^-}),$$

or with an identity of F_1,

$$I_2^{(\alpha)} = \mathcal{D}(1)^\alpha F_1(1, -\alpha, -\alpha; 2; \frac{1}{1-\gamma^+}, \frac{1}{1-\gamma^-})$$

This function is analytic except for the case $\gamma^\pm = 0, 1, \infty$ and $\gamma^+ = \gamma^-$ (F_1 may regular at these point. It depends on the value of parameter α).

⇒ need special care for the numerical calculation around singular points.

⇒ it can be implemented in a numerical library.

2-point function

- Identity : $F_1 \Rightarrow F$.

For $\gamma^+ = \gamma^-$ or $\gamma^+ \neq 0, 1$:

$$I_2^{(\alpha)} = \frac{\gamma^-}{\alpha+1}\mathcal{D}(0)^\alpha F(1, -\alpha; \alpha+2; \frac{\gamma^-}{\gamma^+}) + \frac{1-\gamma^-}{\alpha+1}\mathcal{D}(1)^\alpha F(1, -\alpha; \alpha+2; \frac{1-\gamma^-}{1-\gamma^+})$$

For $\gamma^+ = 0, 1$ and $\gamma^+ \neq \gamma^-$:

$$I_2^{(\alpha)} = \frac{\gamma^-}{\alpha+1}\mathcal{D}(0)^\alpha F(\alpha+1, -\alpha; \alpha+2; \frac{\gamma^-}{\gamma^- - \gamma^+}) + \frac{1-\gamma^-}{\alpha+1}\mathcal{D}(1)^\alpha F(\alpha+1, -\alpha; \alpha+2; \frac{1-\gamma^-}{\gamma^+ - \gamma^-}).$$

2.2 Massless case

- Case 1: $m_2^2 = 0, m_1^2 \neq 0, m_1^2 \neq p^2$

$$\gamma^- = 0, \quad \gamma^+ = \frac{p^2 - m_1^2}{p^2} \neq 0, 1,$$

$$I_2^{(\alpha)} = \frac{(m_1^2 - p^2)^\alpha}{1+\alpha} F(\alpha+1, -\alpha; \alpha+2; \frac{p^2}{p^2 - m_1^2})$$

- Case 2: $m_2^2 = 0, m_1^2 = 0$: $\gamma^+ = 1$ for the previous case

$$I_2^{(\alpha)} = \frac{\Gamma(\alpha+1)^2}{\Gamma(2\alpha+2)} (-p^2)^\alpha.$$

2-point function

- Case 3: $m_2^2 = 0, m_1^2 = p^2$

$$\gamma = \gamma^- = \gamma^+ = 0,$$

$$I_2^{(\alpha)} = \frac{(p^2)^\alpha}{1 + 2\alpha}.$$

2.3 Limit of $d \to 4$

- Expansion in terms of $\alpha = -\epsilon > 0$ in the regular case. With

$$F(1, \epsilon, 2-\epsilon; z) = \frac{1-\epsilon}{1-2\epsilon}\left[\frac{1+z}{2z} - \frac{(1-z)^{1-2\epsilon}}{2z} - \epsilon^2 \frac{1-z}{z}\,\mathrm{Li}_2(z)\right] + \mathcal{O}(\epsilon^3).$$

$$I_2^{(\alpha)} = \frac{1}{1-2\epsilon}\Big[\frac{\gamma^+ + \gamma^-}{2}\mathcal{D}(0)^{-\epsilon} + \frac{2-\gamma^+ - \gamma^-}{2}\mathcal{D}(1)^{-\epsilon}$$
$$- \frac{\gamma^+ - \gamma^-}{2}\left(\mathcal{D}(0)\left\{\frac{\gamma^+ - \gamma^-}{\gamma^+}\right\}^2\right)^{-\epsilon} + \frac{\gamma^+ - \gamma^-}{2}\left\{\mathcal{D}(1)\left(\frac{\gamma^+ - \gamma^-}{1-\gamma^+}\right\}^2\right)^{-\epsilon}\Big]$$
$$+ \mathcal{O}(\epsilon^2)$$

2.4 Summary of the 2-point function

1. General scalar integration of 2-point function is naturally expressed by F_1 (same integral representation of the functions).
2. Locations of singularities are at most $\gamma^{\pm} = 0, 1, \infty$ or $\gamma^{+} = \gamma^{-}$.
3. F_1 has several representations, which are transformed each other with its identities.
4. Some representations are singular at some limit (e.g. massless case). However, we could select a regular form.
5. For the limit of $d \to 4$, F_1 or F reduces to a combination of poly-logarithmic functions.

6. Tensor integration is obtained by the differentiation in terms of mass parameters, even when particles are massless.

7. It will be possible develop a numerical package, which select appropriate representation dynamically looking at the value of parameters.

3 3-point function

■ With $\alpha \geq -1 - \epsilon > -1$,

$$I_3^{(\alpha)} = \int_{x_1,x_2>0,x_1+x_2<1} dx_1 \, dx_2 \, \mathcal{D}^\alpha .$$

where, $\mathcal{D}$ is a quadratic form of x_1 and x_2. The term x_2^2 can be eliminated by the shift: (change variables $(x_1, x_2) \to (x_2, y)$)

$$x_1 = y - r x_2, \quad y = x_1 + r x_2$$

with adjusting of the value of r (projective transformation; tHooft-Vertman '79).

3-point function

As $\mathcal{D}$ is linear in x_2, integration is trivial for x_2. The resulting integration becomes the form:

$$I_3^{(\alpha)} \propto \int \frac{\mathcal{D}^{\alpha+1}}{ay+b}\,dy.$$

Coefficient of x_2 in $\mathcal{D} \Rightarrow$ the denominator (linear in y).

$\mathcal{D}$ can be expressed as a product of linear factors of y.

$$I_3^{(\alpha)} \propto \int \frac{1}{ay+b}\left(1-\frac{y}{\gamma^+}\right)^{\alpha+1}\left(1-\frac{y}{\gamma^-}\right)^{\alpha+1}\,dy.$$

$\Rightarrow$ Lauricella's F_D.

3-point function

■ Lauricella's F_D

This function is a generalization of F, F_1 with n variables.

Its integral representation is

$$F_D(\alpha, \beta_1, \cdots, \beta_n; \gamma; x_1, \cdots, x_n) =$$

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1} \prod_{i=1}^{n} (1-x_i t)^{-\beta_i} dt$$

I found the following identity:

$$z^{p-1}(1-z)^{q-1} \prod_{i=1}^{n} (1-x_i z)^{-\beta_i} = \frac{d}{dz} \frac{z^p}{p} F_D\left(p, (\beta_i), 1-q; p+1; (x_i z), z\right).$$

⇒ the primitive function of a product of linear factor with arbitrary power is expressed by F_D.

3-point function

■ 3-point function is expressed by F_D.

The integration domain becomes slightly complicated by the projective transformation. It is handled systematically with exterior derivative and Stokes' theorem.

The result is

$$I_3^{(\alpha)} = \frac{1}{\alpha+1}\sum_{k=0}^{2}\frac{\mathcal{D}_k(0)^{\alpha+1}}{a_1}\frac{d_{k,1}}{d_{k,0}}F_D(1,1,-\alpha-1,-\alpha-1;2;-\frac{d_{k,1}}{d_{k,0}},\frac{1}{\gamma_k^+},\frac{1}{\gamma_k^-})$$

where, $d_{k,j}$ is brought from the parameterization of the boundary after the projective transformation.

■ For these special values of parameters, F_D reduces to F_1.

3-point function

- F_D with n variables is expressed by two F_Ds with $n-1$ variables:

$$
\begin{aligned}
&F_D(1,(\beta_i)_{i<n},\beta_n;\ 2;\ (x_i)_{i<n},x_n) \\
&\quad = \frac{1}{1-\beta_n}\prod_{i=1}^{n-1}\left(\frac{x_n-x_i}{x_n}\right)^{-\beta_i}\frac{1}{x_n} \\
&\qquad \times F_D(1-\beta_n,\ (\beta_i)_{i<n};\ 2-\beta_n;\ \left(\frac{x_i}{x_i-x_n}\right)_{i<n}) \\
&\qquad - \frac{1}{1-\beta_n}\prod_{i=1}^{n-1}\left(\frac{x_n-x_i}{x_n}\right)^{-\beta_i}\frac{(1-x_n)^{1-\beta_n}}{x_n} \\
&\qquad \times F_D(1-\beta_n,\ (\beta_i)_{i<n};\ 2-\beta_n;\ \left(\frac{x_i(1-x_n)}{x_i-x_n}\right)_{i<n}).
\end{aligned}
$$

Expansion in terms of ϵ for 4-dimension $\Rightarrow$ F_1 reduces to F and then to poly-logarithmic functions, as in the case of 2-point function.

4 4-point function

- 4-point function:

$$I_4^{(\alpha)} = \int_{\mathbb{R}_{\geq 0}} d^4x \, \delta\left(1 - \sum_{j=1}^{4} x_j\right) \mathcal{D}^\alpha.$$

$\mathcal{D}$ is a homogeneous quadratic form of x_j.

For scalar integration, $\alpha = -2 - \epsilon$ $(\epsilon < 0)$.

4-point function

4.1 First two integrations

■ Projective transformation

⇒ $\mathcal{D}$ becomes a linear function of a new variable y. The first integration is easy.

$$\int \mathcal{D}^{\alpha} dy = \frac{1}{\alpha + 1} \frac{\mathcal{D}^{\alpha+1}}{\partial_y \mathcal{D}}.$$

$\mathcal{D}$ is quadratic in terms of other variables, while $\partial_y \mathcal{D}$ is linear factor.

4-point function

■ Projective transformation once more

⇒ $\mathcal{D}$ becomes a linear function of a new variable.

⇒ Integrate with the formula (special case for F_D):

$$\beta \frac{z^{\beta-1}}{1-z} = \frac{d}{dz} z^{\beta} F(1, \beta; \beta+1; z),$$

where F is Gauss' hypergeometric function. It is possible to select the variable z such that $z \propto \mathcal{D}$ and $1 - z \propto \partial_y \mathcal{D}$.

⇒ The second integration is expressed with F.

Integration domain : modified by changing variables and divided for simplification of its shape.

⇒ exterior derivative ⇒ Stokes' theorem.

4-point function

After the integration twice:

$$I_4^{(\alpha)} = \sum_{k=1}^{3} \sum_{\ell=1,\ell\neq k}^{4} \xi_k^{(4)} \xi_\ell^{(k)} \int_{L_{k\ell}} [g_k + h_k(e_k)]\, dy_{k\ell},$$

$$g_k = \frac{1}{(\alpha+1)(\alpha+2)} \frac{e_k^{\alpha+1}}{d_k^{\alpha+2}} \left(\frac{d_k \mathcal{D}_k}{e_k}\right)^{\alpha+2} F(1, \alpha+2, \alpha+3; \frac{d_k \mathcal{D}_k}{e_k}),$$

where

- $\mathcal{D}_k$, e_k: quadratic form of integration variable, d_k: constant.
- ξ^k: 4-dimensional vector ⇐ projective transformation.
- $L_{k\ell}$: line segment ⇐ boundary of the integration domain.
- h_k: arbitrary function ⇐ integration constant.

4.2 Third integration: regular and 4-dimensional case

Scalar integration without IR divergence: $\alpha = -2 - \epsilon, \ \epsilon \to -0$

$$g_k = \frac{1}{\epsilon(1+\epsilon)} \frac{\mathcal{D}_k{}^{-\epsilon}}{e_k} F(1, -\epsilon, 1-\epsilon; \frac{d_k \mathcal{D}_k}{e_k})$$

Expansion in terms of ϵ

$$g_k = \frac{1}{\epsilon(1+\epsilon)} \frac{1}{e_k} [1 - \epsilon \log \mathcal{D}_k] \left[1 + \epsilon \log \frac{e_k - d_k \mathcal{D}_k}{e_k}\right] + O(\epsilon)$$

$$= \frac{1}{\epsilon(1+\epsilon)} \frac{1}{e_k} \left[-1 - \epsilon \log \mathcal{D}_k + \epsilon \log \frac{e_k - d_k \mathcal{D}_k}{e_k}\right] + O(\epsilon)$$

4-point function

The pole term of $1/\epsilon$ can be eliminated by taking $h_k(e_k)$ as:

$$h_k(e_k) = \frac{1}{\epsilon e_k} \left[1 + \epsilon \log e_k\right].$$

We have:

$$I_4 = \sum_{k=1}^{3} \sum_{\ell=1,\ell\neq k}^{4} \xi_k^{(4)} \xi_\ell^{(k)} \int_{L_{k\ell}} \frac{1}{e_k} \log \frac{e_k - d_k \mathcal{D}_k}{\mathcal{D}_k} \, dy_{k\ell} + \mathcal{O}(\epsilon).$$

When quadratic factors are represented by products of linear factors, the expression is becomes a linear combination of the following terms.

$$\int_0^1 \frac{\log(x-a)}{x-b} \, dx.$$

This is integrated out with Li_2 and $\log$.

4.3 Third integration: general case

4-point function with the last integration:

$$I_4^{(\alpha)} = \sum_{k=1}^{3} \sum_{\ell=1,\ell\neq k}^{4} \xi_k^{(4)} \xi_\ell^{(k)} \int_{L_{k\ell}} [g_k + h_k(e_k)] \, dy_{k\ell},$$

$$g_k = \frac{1}{(\alpha+1)(\alpha+2)} \frac{e_k^{\alpha+1}}{d_k^{\alpha+2}} \left(\frac{d_k \mathcal{D}_k}{e_k}\right)^{\alpha+2} F(1, \alpha+2, \alpha+3; \frac{d_k \mathcal{D}_k}{e_k})$$

with $\mathcal{D}_k$ and e_k quadratic in y_{kl}.

Problem : F appears in the integrand.

$\Rightarrow$ partial integration.

With recursion relation of F, g_k can be expressed by:

$$g_k = \frac{1}{(\alpha+1)(\alpha+2)} \frac{e_k^{\alpha+1}}{d_k^{\alpha+2}} \left(\frac{d_k \mathcal{D}_k}{e_k}\right)^{\alpha+2} + \frac{1}{(\alpha+1)(\alpha+3)} \frac{e_k^{\alpha+1}}{d_k^{\alpha+2}} \left(\frac{d_k \mathcal{D}_k}{e_k}\right)^{\alpha+3} F(1, \alpha+3, \alpha+4; \frac{d_k \mathcal{D}_k}{e_k})$$

Factor $e_k^{\alpha+1}$ is integrable:

$$e_k^{\alpha+1}(x) = \frac{df(x)}{dx}, \qquad e_k(x) = e_k(0)\,(1 - \frac{x}{w_5})(1 - \frac{x}{w_6}),$$

$$f(x) = \frac{1}{\alpha+2} \frac{e_k^{\alpha+2}(x)}{\tilde{e}_k(w_6 - w_5)} \left[1 - 2F(-\alpha-2, \alpha+2, \alpha+3; \frac{w_5 - x}{w_6 - x})\right].$$

Partial integration:

$$I_4^{(\alpha)} = \sum_{k=1,k\neq m}^{4} \sum_{\ell=1,\ell\neq k}^{4} \xi_k^{(m)} \xi_\ell^{(k)} [J_1 + J_2 + J_3],$$

$$J_1 = \frac{1}{(\alpha+1)(\alpha+2)} \int_0^1 \frac{\mathcal{D}_k{}^{\alpha+2}}{e_k}\, dx,$$

$$J_2 = \frac{1}{(\alpha+1)(\alpha+3)} \frac{1}{d_k^{\alpha+2}} \Big[f(x) \left(\frac{d_k \mathcal{D}_k}{e_k}\right)^{\alpha+3} F(1, \alpha+3, \alpha+4; \frac{d_k \mathcal{D}_k}{e_k}) \Big]_{x=0}^{1},$$

$$J_3 = -\frac{1}{\alpha+1} \frac{1}{d_k^{\alpha+2}} \int_0^1 f(x) \left(\frac{d_k \mathcal{D}_k}{e_k}\right)^{\alpha+2} \Big\{ \frac{d}{dx} \log\left[\frac{e_k - d_k \mathcal{D}_k}{e_k}\right] \Big\}\, dx.$$

Here we used the following identity:

$$\frac{d}{dx} R(x)^{\alpha+3} F(1, \alpha+3, \alpha+4; R(x)) = -(\alpha+3) R(x)^{\alpha+2} \frac{d}{dx} \log[1 - R(x)].$$

J_1 is integrable with F_D. J_2 is a product of F. f in J_3 is the problem.

Investigating the limit of 4-dimensional space-time $\alpha = -2 - \epsilon \to -2$, one can confirm that integration in J_3 does not produce new pole of $1/\epsilon$.

$\Rightarrow$ expansion in terms of ϵ in the integrand.

$$F(\epsilon, -\epsilon, 1 - \epsilon; z) = 1 + \mathcal{O}(\epsilon^2).$$

$$J_3 = \frac{1}{\epsilon(1+\epsilon)} \frac{1}{\tilde{e}_k(w_6 - w_5)} \int_0^1 \mathcal{D}_k{}^{\alpha+2} \frac{d}{dx} \log\left[\frac{e_k - d_k \mathcal{D}_k}{e_k}\right] + \mathcal{O}(\epsilon).$$

Since factor $d \log / dx$ is expressed by a sum of inverse of linear term of x, J_3 is expressed by F_D.

$\Rightarrow I_4^{(\alpha)} =$ (combination of F and F_D) $+ \mathcal{O}(\epsilon)$.

5 Sample numerical calculation

F and F_D have many parameters and variables

$\Rightarrow$ it is hard to construct general numerical package to calculate them.

$\Leftarrow$ we need values for some special combination of parameters.

■ Sample numerical calculation

(scalar integration: cases of Duplančić and Nižić)

- All particles are massless.
- At least one external particle is on-shell ($p_1^2 = 0$).
- They have IR divergences (poles in terms of ϵ).
- Calculate up to $\mathcal{O}(\epsilon^0)$.
- Calculate tensor integrations up to rank $= 4$.

Numerical calculation

- Loop integration:
  - exactly integrable with F_D.
    - 4 or 3 particles are on-shell
    - "easy case" of 2 particles are on-shell (diagonal external particles of the box diagram are on-shell)
  - integrable with F_D up to $\mathcal{O}(\epsilon^0)$ ($\Leftarrow$ partial integration).
    - "hard case" of 2 particles are on-shell (two adjacent external particles of the box diagram are on-shell),
    - 3 particles are on-shell

5.1 Reduction of F_D to F

F_D can be rearranged into the following form (using identities if necessary):

$$F_D(j_1+1, j_3+1, j_2, j_5, j_5; j_1+j_4+j_5-\epsilon; z_1, z_2, z_3, z_4), \qquad (j_k \in \mathbb{Z}).$$

$\Rightarrow$ Reduction of F_D to F with identities of F_D.

$\Rightarrow$ Recursion relations of the parameters.

$\Rightarrow$ Expansion in terms of ϵ.

■ The result contains:

$$F(\epsilon, -\epsilon, 1-\epsilon; z),$$

$$F(1, 1-a\epsilon, 1-b\epsilon; z), \qquad (a, b \in \mathbb{Z}, a \neq b).$$

5.2 Expansion of F

■ Expansion formula of F in terms of ϵ:

$$F(\epsilon, -\epsilon, 1-\epsilon; z) = 1 - \epsilon^2 \operatorname{Li}_2(z) + O(\epsilon^3),$$

$$F(1, 1-a\epsilon, 1-b\epsilon; z) = -\frac{(1-z)^{-(b-a)\epsilon}}{z}\left[1 + (b-a)b\epsilon^2 \operatorname{Li}_2(z)\right] + O(\epsilon^3).$$

The box integration in these examples up to finite order can be written with $\log$ and Li_2

$\Rightarrow$ numerical calculable.

Numerical calculation

■ The expression is not necessary numerical stable when it is written with $\mathrm{Li}_2(z)$.

For example, there appears $F(1, m-\epsilon, m+1-\epsilon; z),\ (m \geq 1)$ in the tensor integration. When this function is expanded in terms of ϵ, coefficients of the expansion includes:

$$\frac{1}{z^m}\left[\mathrm{Li}_k(z) - \sum_{j=1}^{m-1} \frac{z^j}{j^k}\right] \sim \mathcal{O}(z^0).$$

First $m-1$ terms cancel out in the power series expansion of Li_{k+1} around $z \sim 0$, and then cancels with the denominator.

⇒ It is better to calculate by power series expansion of $F(1, m-\epsilon, m+1-\epsilon; z)$ directly around z .

Numerical calculation

We have been developing a library

1. Entry points of subroutines are F_D or F for limited conditions of parameters.

2. These subroutines return an array of coefficients of $1/\epsilon^2, 1/\epsilon, 1, \epsilon, \cdots$ up to necessary and calculable order.

   (At this moment, they are prepared only for this sample calculation.)

3. Inside of the subroutines, appropriate identities or calculation methods are selected in looking at the values of parameters and variables.

5.3 Comparison

We have constructed a library to see whether this method works well or not for our example.

Two programs are prepared for the confirmation:

- “program-1”: The calculation with library of F_D and F.
- “program-2”:
  - The first two integration is calculated analytically.
  - Coefficients of $1/\epsilon^2, 1/\epsilon^1, 1/\epsilon^0$ are expressed as one-dimensional integrations.
  - The last integration is calculated numerically (Romberg method)

Numerical calculation

■ We have compared among “program-1”, “program-2” and “Golem” package up to rank $= 4$ at 7560 points for the parameters:

$$p_1^2 = 0, \qquad p_2^2 = 0, \pm 50, \qquad p_3^2 = 0, \pm 55, \qquad p_4^2 = 0, \pm 60$$

$$s = \pm 200, \qquad t = \pm 123$$

$$n_i = 0, 1, 2, 3, 4, \qquad \sum_i n_i \leq 4 \qquad \text{(rank of tensor integration)}$$

Numerical calculation

■ The maximum relative errors (measured by the distance on the complex plane) among these points are:

		maximum error
program-1(d)	program-2(d)	7.65×10^{-7}
program-1(d)	Golem(d)	9.13×10^{-10}
program-1(d)	program-1(q)	3.98×10^{-10}
Golem(d)	Golem(q)	5.17×10^{-10}
program-1(q)	Golem(q)	1.38×10^{-18}

(d) : double precision, (q) : quadruple recision.

Accuracy of the library at this moment will be similar to Golem package.

6 Summary

- 2-, 3-point functions are expressed with F_D, exactly for any combination of physical parameters and any space-time dimensions.
- 4-point functions are expressed with F_D, up to $\mathcal{O}(\epsilon^0)$ for any combination of physical parameters.
- Many identities of F_D are developed for these calculations.
- A program library of F and F_D is under developing.
- Sample numerical calculations for massless QCD with IR divergences agree with Golem package.
- 4-point function seems not to be integrated with F_D.

  It will need more general hypergeometric functions (numerical calculations?)