Numerical approach to calculation of Feynman loop integrals

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Loop integrals

$$(-1)^N \left(\frac{1}{4\pi}\right)^{nL/2} \Gamma(N-nL/2)\int_0^1 \prod_{i=1}^{N} dx_i \delta(1-x_1 \cdots - x_N) \frac{C^{N-n(L+1)/2}}{(D-i\varepsilon C)^{N-nL/2}}$$

$n = 4$

L	N	$C^{N-2(L+1)}/D^{N-2L}$	
1	3	$1/(D-i\varepsilon)$	one-loop vertex
	4	$1/(D-i\varepsilon)^2$	one-loop box
	5	$1/(D-i\varepsilon)^3$	one-loop pentagon
2	5	$1/C(D-i\varepsilon C)$	two-loop selfenergy
	6	$1/(D-i\varepsilon C)^2$	two-loop vertex
	7	$C/(D-i\varepsilon C)^3$	two-loop box

What is Direct computation method ?

E. De Doncker, Y.Shimizu J.Fujimoto, FY CPC 159('04)145.

In the analytic treatment, ε in the denominator is an infinitesimal number (in complex analysis) while we consider it a finite (sometimes rather large) number

$$I = \lim_{\varepsilon \to 0} \int_0^1 dx \int_0^{1-x} dy \frac{1}{D(x,y) - i\varepsilon}$$

Sometimes denominator becomes 0 in the integration region.

Change ε as $\varepsilon_l = \dfrac{\varepsilon_0}{c^l}, \quad c > 1$

Do integration

and get I(ε_l), $\varepsilon_l > 0$, $l = 0,1,2,\ldots$

$$\Re(I(\varepsilon_l)) = \int_0^1 dx \int_0^{1-x} dy \frac{D(x,y)}{D(x,y)^2 + \varepsilon_l^2},$$

$$\Im(I(\varepsilon_l)) = \int_0^1 dx \int_0^{1-x} dy \frac{\varepsilon_l}{D(x,y)^2 + \varepsilon_l^2}$$

Extrapolate I(ε_l)

and get the result when ε becomes 0

using ε-algorithm by P.Wynn.

$$\Re(I) = \lim_{\varepsilon \to 0} \{\Re(I(\varepsilon_l))\},$$

$$\Im(I) = \lim_{\varepsilon \to 0} \{\Im(I(\varepsilon_l))\}$$

Questions about
Direct computation method

How many legs does it handle?

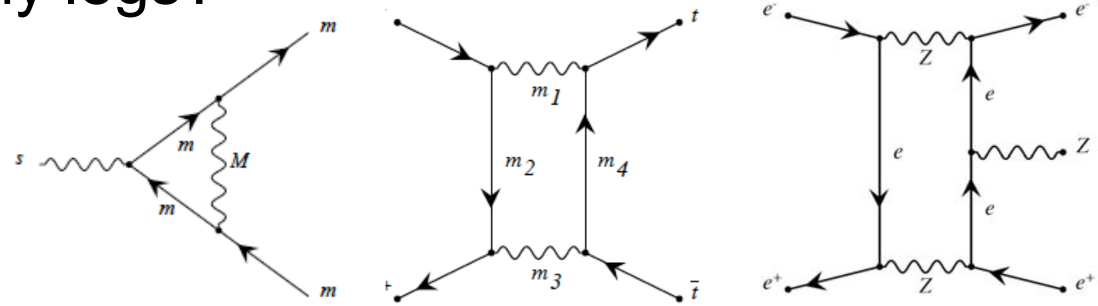
Does it handle two-loop integral?

Is it available in gigantic calculation?

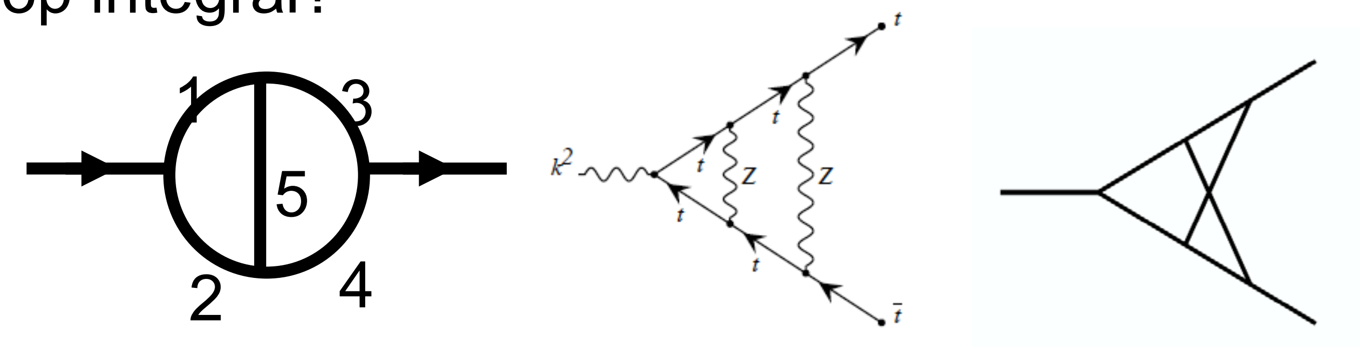
How about the precision?

Experiences

How many legs?



Two-loop integral?



- Comput. Phys. Comm. 169 (2004) 145.
- Nucle. Instr. Meth. in Phys. A 534 (2004) 269.
- PoS (ACAT) 085 in 2007
- PoS (ACAT08) 122 in 2008
- PoS(ACAT2010)073 in 2010
- ….

Numerical results show good agreements to ones calculated by different methods.

Table 10
S^{221} compared with the expansions described in Appendix F f
The sum over n in Eq. (F.4) is restricted to $n \leqslant 20$

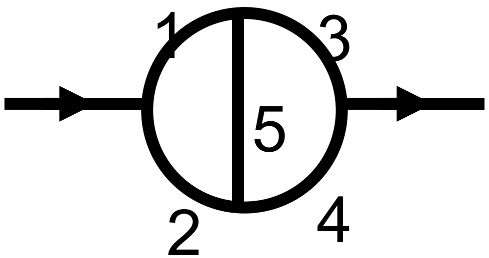
G. Passarino, S. Uccirati / Nuclear Physics B 629 (2002) 97–187

s [GeV^2]	Re S^{221}	Re S^{221} expansion	Im S^{221}	Im S^{221} expansion
0.1	0.0496(2)	0.049542	0.01566(3)	0.015709
0.5	0.04154(5)	0.041506	0.015722(8)	0.0157145
1.0	0.03805(3)	0.038051	0.015717(9)	0.0157211
5.0	0.030085(9)	0.0300729	0.015776(4)	0.0157739
10.0	0.026682(9)	0.0266732	0.015836(6)	0.0158406

Our Result : Real.Part

0.1: 0.049542813200543447 0.320x10^{-08} time= 15597 sec
0.5: 0.041505757995489756 0.103x10^-{09} time= 20099 sec
1.0: 0.038051319819241355 4 0.315x10^-{10} time= 23477 sec
5.0: 0.030072881454860145 3 0.113x10^-{10} time= 49088 sec
10.0: 0.026673228264904418 1 0.501x10^{-10} time= 70116 sec

$$m_1^2 = m_2^2 = m_5^2 = 100,$$
$$m_3^2 = m_4^2 = 0$$



Our Result: Imag. Part

0.1: 0.015709276263800517 3 0.578x10^{-08} time= 8104 sec
0.5: 0.015714513291842112 4 0.678x10^{-09} time= 8569 sec
1.0: 0.015721070842473792 3 0.760x10^{-10} time= 9423 sec
5.0: 0.015773850653579793 4 0.796x10^{-10} time= 23233 sec
10.0: 0.015840637053487770 9 0.315x10^{-10} time= 36018 sec

From experiences

- It can handle arbitrary masses (real masses and complex masses)
- It can handle divergences as
  - divergence when the denominator becomes zero
  - Infrared divergence
  - (Ultraviolet divergence)
- For gigantic calc.
  - Speed control
  - Precision control

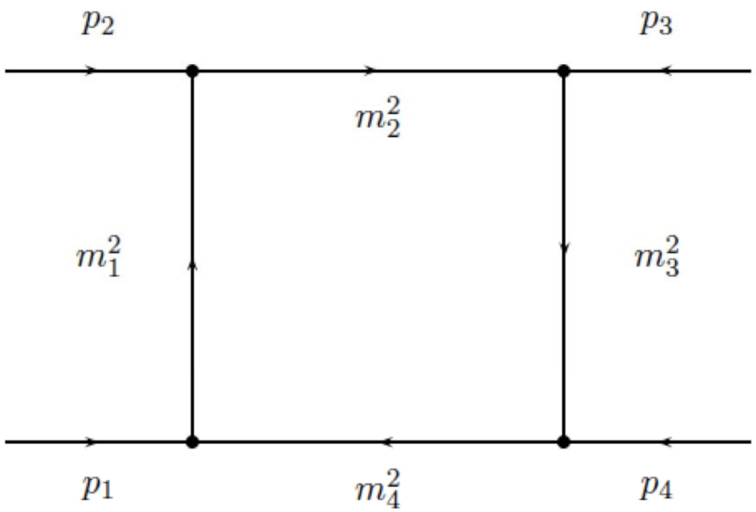
57.3TFLOPS since 2006 in KEK
http://scwww.kek.jp/index_e.html

GRAPE-DR model450,
PCI Express card

- One-loop diagram with complex masses
  - ➢work with Vietnamese group in early summer 2010

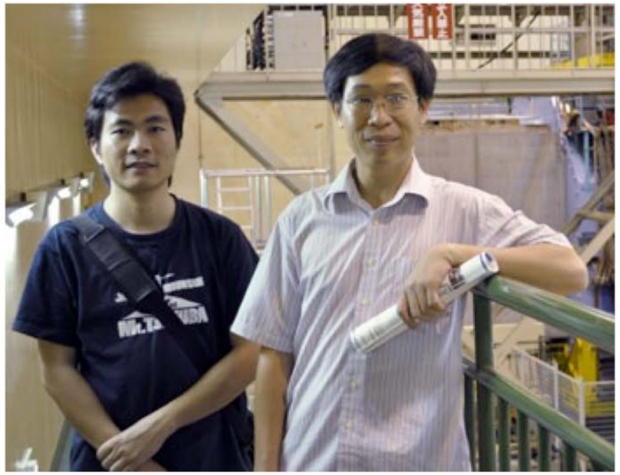
- Recent progress since October 2009
  - ➢Two-loop box integral

one-loop integral with complex mass



Input parameters are

$$
\begin{aligned}
p_1^2 &= 10.0 &,\quad m_1^2 &= 10.0 - 5.0i; \\
p_2^2 &= -60.0 &,\quad m_2^2 &= 20.0 - 0.0i; \\
p_3^2 &= -10.0 &,\quad m_3^2 &= 10, 100, 1000, 10000; \\
p_4^2 &= -10.0 &,\quad m_4^2 &= 40.0 - 10.0i; \\
s = (p_1 + p_2)^2 &= 200.0 &,\quad t = (p_2 + p_3)^2 &= -10.0
\end{aligned}
$$



Khiem san (left) and Son san (right)

Comparison

Hoang *Son* Do, Phan Hong Khiem

Dao Thi Nhung, Le Duc Ninh

m_3^2		XLOOPS-GiNaC	LoopTools 2.4
10	Real	$2.4766077185850360 53E^{-4}$	0.00260538
	Imag:	$4.830551285215682584E^{-4}$	0.00192925
100	Real:	$8.154341559464273185 3E^{-5}$	-0.00058759
	Imag:	$1.545005716305943352 2E^{-4}$	0.000740276
1000	Real:	$1.476917164580908455 25E^{-5}$	$-5.67191e^{-05}$
	Imag:	$2.219790835442331104 7E^{-5}$	$3.75257e^{-05}$
100000	Real:	$1.763183638649841257 6E^{-7}$	$-5.34807e^{-07}$
	Imag:	$2.356524818180093298 9E^{-7}$	$3.28924e^{-07}$

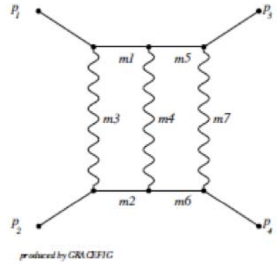
XLOOPS-GiNaC : http://wwwthep.physik.uni-mainz.de/~xloops/
LoopTools: http://www.feynarts.de/looptools/

good agreement !

m_3^2		XLOOPS-GiNaC	LoopTools 2.5	Minami-Tateya Group
10	Real	$2.4766077185850360 53E^{-4}$	0.000247661	$0.2476607718651927D^{-03}$
	Imag:	$4.830551285215682584E^{-4}$	0.000483055	$0.4830551284955444D^{-03}$
100	Real:	$8.154341559464273185 3E^{-5}$	$8.15434e^{-05}$	$0.8154341560206131D^{-04}$
	Imag:	$1.545005716305943352 2E^{-4}$	0.000154501	$0.1545005716281967D^{-03}$
1000	Real:	$1.476917164580908455 25E^{-5}$	$1.47692e^{-05}$	$0.1476917164636933D^{-04}$
	Imag:	$2.219790835442331104 7E^{-5}$	$2.21979e^{-05}$	$0.2219790835451625D^{-04}$

LoopTools is upgraded to 2.5

Two-loop planar box



$$\mathcal{I} = -\int_0^1 dx_1\, dx_2\, dx_3\, dx_4\, dx_5\, dx_6\, dx_7\, \delta(1-\sum_{\ell=1}^{7} x_\ell)\frac{\mathcal{C}}{(\mathcal{D}+i\epsilon\mathcal{C})^3},$$

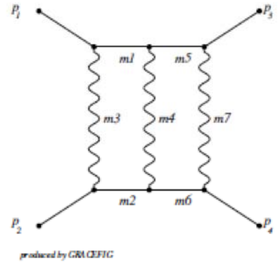
$$\begin{aligned}
\mathcal{D} = & -\mathcal{C}\sum x_\ell m_\ell^2 \\
& + \{s(x_1x_2(x_4+x_5+x_6+x_7)+x_5x_6(x_1+x_2+x_3+x_4)+x_1x_4x_6+x_2x_4x_5) \\
& + tx_3x_4x_7 \\
& + p_1^2(x_3(x_1x_4+x_1x_5+x_1x_6+x_1x_7+x_4x_5)) \\
& + p_2^2(x_3(x_2x_4+x_2x_5+x_2x_6+x_2x_7+x_4x_6)) \\
& + p_3^2(x_7(x_1x_4+x_1x_5+x_2x_5+x_3x_5+x_4x_5)) \\
& + p_4^2(x_7(x_1x_6+x_2x_4+x_2x_6+x_3x_6+x_4x_6))\},
\end{aligned}$$

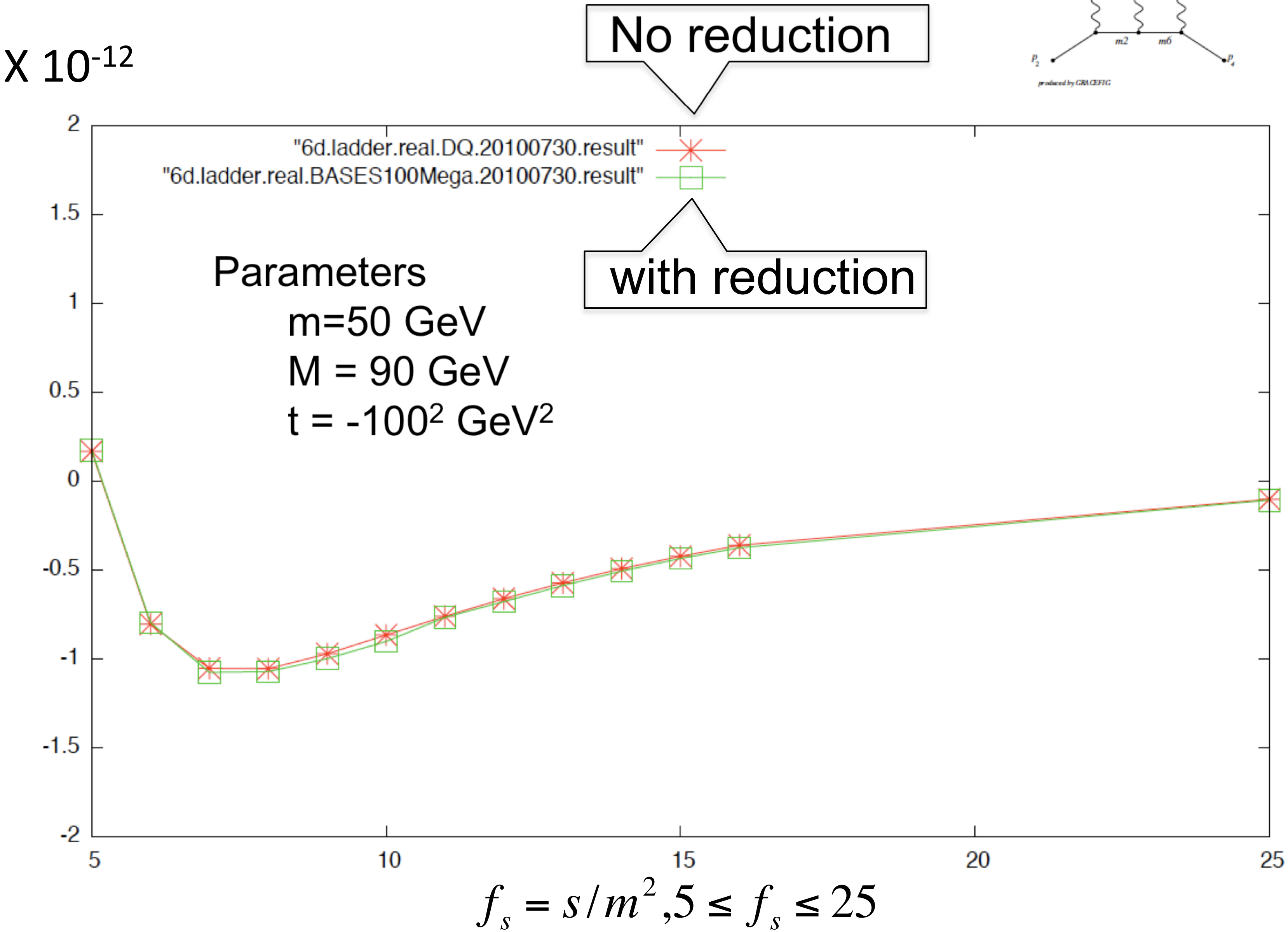
and

$$\mathcal{C} = (x_1+x_2+x_3+x_4)(x_4+x_5+x_6+x_7) - x_4^2.$$

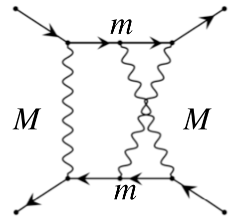
Here, $s=(p_1+p_2)^2=(p_3+p_4)^2$, $\quad t=(p_1+p_3)^2=(p_2+p_4)^2$ and $\; p_1+p_2+p_3+p_4=0$.

Example of direct computation method
two-loop planar box with masses





Two-loop non-planar box



$$\mathcal{I} = - \int_0^1 dx_1\, dx_2\, dx_3\, dx_4\, dx_5\, dx_6\, dx_7\, \delta(1 - \sum_{\ell=1}^{7} x_\ell) \frac{\mathcal{C}}{(\mathcal{D} + i\epsilon\mathcal{C})^3},$$

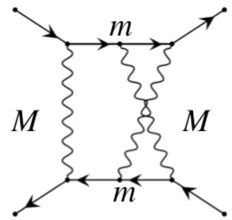
$$\begin{aligned}
\mathcal{D} &= -\mathcal{C} \sum x_\ell m_\ell^2 \\
&+ \{s(-x_1x_5x_6 + x_2x_3x_4 + x_2x_3x_5 + x_2x_3x_6 + x_2x_3x_7 + x_2x_6x_7 + x_3x_4x_5) \\
&+ t(x_1(x_4x_7 - x_5x_6)) \\
&+ p_1^2(x_1(x_2x_4 + x_2x_5 + x_2x_6 + x_2x_7 + x_4x_5 + x_5x_6)) \\
&+ p_2^2(x_1(x_3x_4 + x_3x_5 + x_3x_6 + x_3x_7 + x_5x_6 + x_6x_7)) \\
&+ p_3^2(x_1x_5x_6 + x_1x_5x_7 + x_2x_4x_7 + x_2x_5x_7 + x_3x_5x_6 + x_3x_5x_7 + x_4x_5x_7 + x_5x_6x_7) \\
&+ p_4^2(x_1x_4x_6 + x_1x_5x_6 + x_2x_4x_6 + x_2x_5x_6 + x_3x_4x_6 + x_3x_4x_7 + x_4x_5x_6 + x_4x_6x_7)\}
\end{aligned}$$

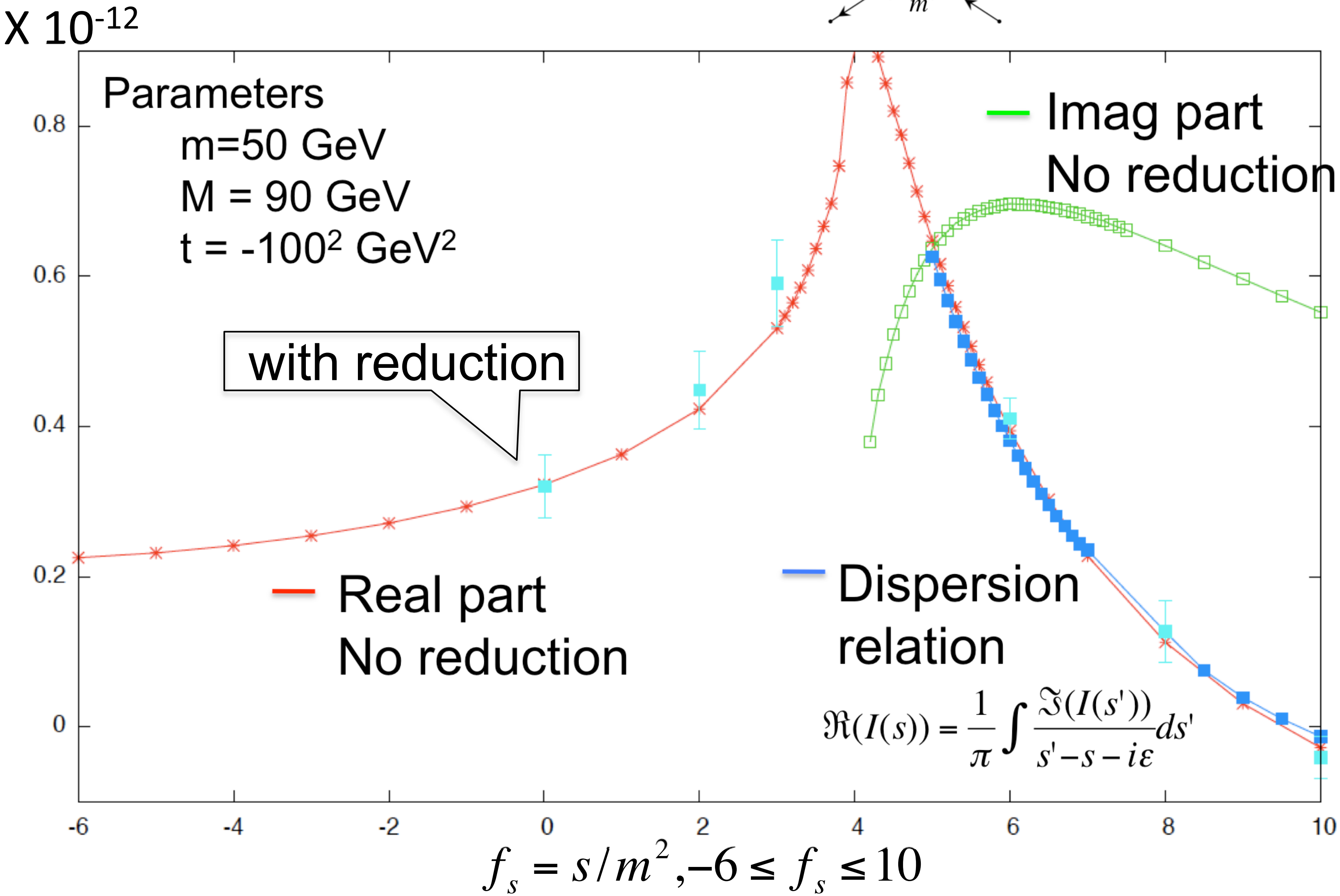
and

$$\mathcal{C} = (x_1 + x_2 + x_3 + x_4 + x_5)(x_1 + x_2 + x_3 + x_5 + x_6 + x_7) - (x_1 + x_2 + x_3)^2.$$

Here, $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$, $t = (p_1 + p_3)^2 = (p_2 + p_4)^2$ and $p_1 + p_2 + p_3 + p_4 = 0$.

Example of direct computation method
two-loop non-planar box with masses





Summary

- We develop Direct computation method, a fully numerical computation method for loop integrals.
- Direct computation method is a combination of multi-dimensional numerical integration and an extrapolation method.
- Direct computation method is available for loop integrals with complex masses.
- Using Direct computation method , we calculated two-loop planar/non-planar integrals with masses.

Thank you!